Proposition: Let \((x_n)\) be a sequence in a B-space \(X\). TFAE

i) \(\sum x_n\) converges unconditionally.

ii) \(\forall n_1 < n_2 < \ldots \ldots \sum x_{n_k}\) converges.

iii) \(\forall \text{ (en)}\) in \([-1,1]\) \(\sum x_n\) converges.

iv) \(\forall \alpha > 0 \exists n \in \mathbb{N}\) s.t. \(\forall F \in C[K] \text{ finite with } \min(f) > \alpha\), \(\|\sum x_n\| < \epsilon\).

Proposition: Let \((x_n)\) be a sequence in a B-space \(X\) so that \(\sum x_n\) converges unconditionally. Then,

(i) For any \(n_1 < n_2 < \ldots \ldots\), \(\sum x_{n_k}\) converges unconditionally.

(ii) If \(\sum_{n=1}^{\infty} x_n = x\), then \(\forall\) permutations \(n_1, n_2, \ldots\), \(\sum_{n=1}^{\infty} x_{n(n)} = x\).

(iii) For any \(a = (a_n) \in \ell^\infty\), \(\sum a_n x_n\) converges. Specifically, \(T: \ell^\infty \to X\) given by \(T(a) = \sum a_n x_n\) is well defined, linear, and bounded.

Remark: If \(I = \{i_n: n \in \mathbb{N}\}\) is a countable index set so that \(\sum x_{i_n}\) converges unconditionally, we may write \(\sum_{i \in I} x_i\) without specifying order, because the limit is unique. Also, \(\forall\) \(J \subseteq I\), \(\sum_{i \in J} x_i\) is well defined.

Proof: (i) is almost obvious from part (iii) of the previous proposition.
For (ii), take $\varepsilon > 0$ and $n_2$ s.t. $\forall n \geq n_2$
\[ \| \sum_{i=1}^{n} x_i - x \| < \varepsilon. \]
Take $n_2 \in \mathbb{N}$ s.t. $\forall F \subset\subset \mathbb{N}$ finite with
\[ \min(F) > n_2 \]
we have $\| \sum_{i \in F} x_i \| < \varepsilon$.
Let $\mu : [1] \rightarrow [1, \ldots, n_2]$ be a permutation. Pick $n_0 \in \mathbb{N}$
s.t. $\{ 1, 2, \ldots, \max\{n_1, n_2\} \} \subset \mu(\{1, \ldots, n_0\})$.
Then, for $\mu \in \mathbb{N}$,
\[ \| \sum_{i=1}^{n_0} x_{\mu(i)} - x \| = \| \sum_{i=1}^{n_0} x_i + \sum_{i \in F(\mu) : n_\mu(n) > n_0 n_2} x_i \|
\leq \| \sum_{i=1}^{n_0} x_i - x \| + \| \sum_{i \in F(\mu)} x_i \| \leq 2\varepsilon \]
because $\min(F(\mu)) > n_0 n_2 > n_2$.

(iii) Set $M = \sup\{a(n)\}$, let $\varepsilon > 0$, and pick $n_0 \in \mathbb{N}$ s.t. $\forall F \subset\subset \mathbb{N}$ finite with
\[ \min(F) > n_0 \]
we have $\| \sum_{n \in F} x_n \| < \varepsilon / 4M$.
Let $n_0 \leq l \leq m$ and evaluate $\| \sum_{n=l}^{m} a(n) x_n \|$. By H-B
\[ x^* \in \mathbb{X}^* \text{ with } \| x^* \| = 1 \quad \& \quad x^*(\sum_{n=l}^{m} a(n) x_n) = \| \sum_{n=l}^{m} a(n) x_n \|. \]
Define $F_1 = \{ l \leq n \leq m : x^*(x_n) > 0 \}, F_2 = \{ l \leq n \leq m : x^*(x_n) < 0 \}$
Then,
\[ \| x^*(\sum_{n=l}^{m} a(n) x_n) \| \leq M \sum_{n \in F_1} | x^*(x_n) | + M \sum_{n \in F_2} | x^*(x_n) | \]
\[ = M \left( \| x^*(\sum_{n \in F_1} x_n) \| + \| x^*(\sum_{n \in F_2} x_n) \| \right) \leq M \left( \sum_{n \in F_1} | x_n | + \sum_{n \in F_2} | x_n | \right) < \varepsilon. \]
Thus, $\sum_{n=1}^{\infty} a(n) x_n$ converges. Even more, observe that
if $M \leq 1$ and we take $\varepsilon = \frac{\varepsilon}{4}$, then for the given $n_0$,
\[ \| \sum_{n=1}^{\infty} a(n) x_n \| \leq \| \sum_{n=1}^{n_0} a(n) x_n \| + \| \sum_{n=n_0}^{\infty} a(n) x_n \| \leq \max\{a(n)\} \sum_{n=1}^{n_0} | x_n | + \frac{\varepsilon}{4} \]
\[ \leq \sum_{n=1}^{n_0} | x_n | + \frac{\varepsilon}{4}. \] Thus, the given $T$ satisfies $\| T \| \leq \sum_{n=1}^{\infty} | x_n | + 1$.\]
Weakly unconditionally Cauchy series:

Def: Let \((x_n)_n\) be a sequence in a Banach space \(X\). The series \(\sum x_n\) is called weakly unconditionally Cauchy, if for every \(x^* \in X^*\), \(\sum_{n=1}^{\infty} |x^*(x_n)| < \infty\).

Remark: If \(\sum x_n\) converges unconditionally, then it is WUC. Indeed, if \(x^* \in X^*\) \(\forall n \in \mathbb{N}\) \(\sum_{n=1}^{\infty} x^*(x_n)\) converges in \(\mathbb{R} = \sum_{n=1}^{\infty} |x^*(x_n)| < \infty\).

Remark: A WUC series may not even converge weakly! Take \(x_n = 1/n\) in \(\mathbb{C}^\infty\) does not converge weakly. For any \(x^* \in (\mathbb{C}^\infty)^*\) in \(\ell_1\) \(\sum_{n=1}^{\infty} |x^*(e_n)| = \sum_{n=1}^{\infty} |a_n| = \|x^*\| < \infty\).

Proposition: Let \((x_n)_n\) be a sequence in a Banach space. TFAE

1. \(\sum x_n\) is WUC
2. \(\exists D > 0\) s.t. \(\forall (x_k)_n\) finite, \(\sum_{n \in \mathbb{N}} x_n\) \(\leq D\).
3. \(\exists C > 0\) s.t. \(\forall a_1, \ldots, a_n \in \mathbb{R}\) \(\sum_{k=1}^{n} a_k x_k\) \(\leq C \max_{1 \leq k \leq n} |a_k|\).

Specifically, \(T : C_0 \to X\) given by \(T \alpha = \sum_{n=1}^{\infty} \alpha(n)x_n\) is well defined and bounded.

Proof: ii) \(\Rightarrow\) i) obviously. For ii) \(\Rightarrow\) i) let \(x^* \in X^*\). For \(N \in \mathbb{N}\), set \(F_1 = \{n \in \mathbb{N} : x^*(x_n) \geq 0\}\), \(F_2 = \{n \in \mathbb{N} : x^*(x_n) < 0\}\). Then, \(\sum_{n=1}^{N} |x^*(x_n)| = |x^*(\sum_{n \in \mathbb{N} \setminus F_1} x_n)| + |x^*(\sum_{n \in \mathbb{N} \setminus F_2} x_n)| \leq 2\|x^*\|D\). Thus, \(\sum_{n=1}^{\infty} |x^*(x_n)| \leq \|x^*\|D\).

For i) \(\Rightarrow\) iii) define the set \(A = \{\sum x_{k} x_{k}, a_1, \ldots, a_n \in (\mathbb{C}^\infty)^{\infty}\}\)
We claim \( A \) is bounded. If \( m \neq 0 \), \( \exists x^* \in X^* \text{ s.t. } \sup \{ |x^*(y)| : y \in A \} = \infty \) (by known lemma).

But for any \( y = \sum_{k=1}^{n} a_k x_k \) in \( A \), \( |x^*(y)| \leq \sum_{k=1}^{n} |a_k| |x^*(x_k)| \leq \sum_{k=1}^{n} |x^*(x_k)| = \sup_{k=1}^{\infty} |x^*(x_k)| \leq \sum_{k=1}^{n} |x^*(x_k)| \),

which is finite because of (i). We deduce that

\[ \forall a_1, \ldots, a_n \in \mathbb{R}, \quad \left\| \sum_{k=1}^{n} a_k x_k \right\| \leq \max_{1 \leq k \leq n} |a_k| \sup_{y \in A} |x^*(y)|. \]

The second part follows directly from extending an operator from \( \langle e_i : i \in \mathbb{N} \rangle \subset C_0 \) to \( C_0 \).

The goal is to prove the following:

**Theorem (Bessaga-Pelczynski):** Let \( X \) be a B-space. TFAE

i) \( X \) does not contain an isomorphic copy of \( C_0 \).

ii) Every WUC series converges unconditionally in norm.

iii) \( \Rightarrow \) i) is obvious because in \( C_0 \), \( \langle e_i : i \in \mathbb{N} \rangle \) is WUC but not convergent.

For i) \( \Rightarrow \) ii) we need the following:

**Theorem:** Let \( X \) be a B-space \& \( T : C_0 \rightarrow X \). TFAE

i) \( T \) is compact (i.e. \( \overline{(B_{C_0})^{\infty}} \subset \text{co}(X) \text{ compact} \))

ii) \( T \) is weakly compact (i.e. \( \overline{(B_{C_0})^{\infty}} \subset \text{co}(X) \text{ weakly compact} \))

iii) \( T \) is strictly singular (i.e. \( \not\exists \text{ VC \& closed \& finite-dim. s.t. } T \text{ is an embedding} \))

iv) If \( y_n = \langle e_i \rangle_{i=n} \subset C_0 \text{ then } \lim_{n} \| T(y_n) \| = 0. \)