Theorem: Let $X$ be a separable $B$ space. Then, there exists a bounded linear onto $T: l_2 \rightarrow X$. Specifically, $T(B_2^0) = B_X^0$. ($B_X^0 = \{x \in X : \|x\| < 1\}$)

Corollary: The space $l_2$ has an uncomplemented subspace.

Proof of Thm: Fix $(x_n)_n$ dense in $B_X$. We define $T: l_2 \rightarrow X$ as follows:\[ Ty = \sum_{n=1}^{\infty} y(n)x_n. \]

Note that $\|Ty\| \leq \sum_{n=1}^{\infty} |y(n)| \|x_n\| \leq \sum_{n=1}^{\infty} |y(n)| = \|y\| \|x\|$, thus $T$ is well defined with $\|T\| \leq 1$. To complete the proof, fix $x \in X$ with $\|x\| = \theta < 1$. Pick positive numbers $\varepsilon_2, \varepsilon_3, \varepsilon_4, \ldots$ s.t. $\frac{\varepsilon_2}{\theta} \leq 1 - \theta$. We will inductively choose $n_2 < n_3 < \ldots$ s.t. $\|x - (\theta x_{n_k} + \sum_{i=2}^{k} \varepsilon_i x_{n_i})\| \leq \frac{\varepsilon_k}{\theta}$ for all $k \geq 2$. If we have done this, then set $y \in B_{l_2}$, $y(n) = \begin{cases} 8 & n = n_k, k \geq 2, \\ 0 & \text{otherwise} \end{cases}$. And $Ty = x$.

Because $\frac{x}{\theta} \in B_{l_2}$, for $n_k$ s.t. $\|x - n_k\| \leq \frac{\varepsilon_k}{\theta}$, we have $\|n_k - x\| \leq \frac{\varepsilon_k}{\theta}$. Therefore, $\|Ty\| = \|x\|$. Therefore, $T$ is the desired linear operator.
Assume we have picked $n_1 < \ldots < n_k$. Then, because $\frac{1}{\varepsilon_{k+1}} \| x - (\varepsilon_{k+1} x_{n_1} + \sum_{i=1}^{k} \varepsilon_i x_{n_i}) \| \leq \varepsilon_{k+1}$, if $n_{k+1} > n_k$

s.t. $\| \frac{1}{\varepsilon_{k+1}} (x - (\varepsilon_{k+1} x_{n_1} + \sum_{i=1}^{k} \varepsilon_i x_{n_i})) - x_{n_{k+1}} \| \leq \frac{\varepsilon_{k+2}}{\varepsilon_{k+1}} \Rightarrow$

$\| x - (\varepsilon_{k+1} x_{n_1} + \sum_{i=1}^{k+1} \varepsilon_i x_{n_i}) \| \leq \varepsilon_{k+2}$.

**Exercise:** Let $X$ be a separable Banach space. Find $T \in L(\ell^2, X)$ s.t. $T^* \in L(\ell^2, X^*)$ is an isometric embedding.

**The Schur property of $\ell^2$**

Def: A Banach space $X$ has the Schur property if every weakly compact subset of $X$ is weakly compact.

e.g. finite dimensional spaces.

**Exercise:** If $X$ is reflexive infinite dimensional, then $X$ does not have the Schur property.

**Exercise:** The weak basis of $c_0$ is w-null. Then $c_0$ does not have the Schur property.

**Exercise:** The weak of $\ell^2$ is not w-null. Indeed, take $x_0^* \in (\ell^2)^*$ with $x_0^*(x) = \sum_{n=1}^{\infty} x_n$. This is well defined and $\| x_0^* \| = 1$. If $(e_n)_n$ were w-null then $x_0^*(e_n) \to 0$ but $x_0^*(e_n) = 1$ for $n \in \mathbb{N}$.

**Exercise:** Let $X$ be a Banach space. TFAE

i) $X$ has the Schur property

ii) Every w-null $(x_n)_n$ in $X$ converges to zero in norm.

iii) $I : X \to X$ is completely continuous.
Proof of ii): Let $W \subset X$ be w-compact $\implies W$ is w-closed $\implies$ $W$ is w-1-1-closed. Let $(x_n)_n$ be a seq. in $W$. By E-S it has a w-conv. subseq. $(x_n)_n$ to some $x_0 \in W \implies x_n - x_0 \rightarrow 0 \implies x_n - x_0 \overset{w}{\rightarrow} 0 \implies x_n \overset{w}{\rightarrow} x_0$. Thus $(x_n)_n$ has a convergent subsequence, i.e., $W$ is compact.

Theorem: $\ell_1$ has the Schur property.
Proof: If not, $(x_n)_n$ w-null with $\inf \|x_n\| > 0$. By gliding hump, $(x_n)_n$ has a subseq. equiv to a seminormalized block sequence $\varepsilon$ cont. of $\ell_1$, $\implies \exists T: (x_n) \rightarrow (e_n)_n$ with $T x_n = e_n$, $\implies T$ is w-w cont. $\implies (e_n)_n$ is w-null which it is not.

Remark: If $X$ has the Schur property & $Y \subset X \implies Y$ has the Schur prop.

Theorem: If $X$ has the Schur property & dim$(X) = \infty$, then $\ell_1 \subset X = \exists Y \subset X$ inf. dim. $\ell_1 \subset X$.
we don't have the tools to prove this.

Def: Let $X$ be a B-space.
i) A seq. $(x_n)_n$ in $X$ is called w-Cauchy if $\forall x^* \in X$
$\lim_{n \rightarrow \infty} x^*(x_n)$ exists.
ii) $X$ is called w-sequentially complete if every w-Cauchy sequence in $X$ is w-convergent.

Exercise: Reflexive spaces are wsc. $\ell_1$ is not wsc (since $(\sqrt{n})_n$ is w-candy but not w-convergent).
Proposition: Let $X$ be a Banach space with the Schur property. Then $X$ is wsc. In particular, $\ell_1$ is wsc.

Proof: Let $(x_n)_n$ be a w-Cauchy sequence in $X$.

Claim: $(x_n)_n$ is w-Cauchy & thus convergent, i.e., also w-convergent.

If not, then $\exists$ subseq. $(x_n)_n$ & $\exists \varepsilon > 0$ s.t.

\[\forall n \, \|x_n - x_m\| \geq 2\] \[\text{But } \lim_{n \to \infty} x_n - x_m \Rightarrow 0,\]

i.e., by the Schur property $\lim_{n \to \infty} \|x_n - x_m\| = 0$.

A contradiction.