Banach spaces, lecture 13, 9/25/2019

Subspaces of $c_0$, $c_0$ (part 2) & complemented subspaces of $C_0$ & $\ell_p$, $1 \leq p < \infty$ (part 2)

**Theorem:** Let $X = \ell_p$, $1 \leq p < \infty$ or $X = c_0$. Every closed infinite dimensional subspace of $X$ contains a subspace isomorphic to $X$ complemented in $X$.

**Corollary:** The spaces $c_0$, $\ell_p$ don't contain infinite dimensional reflexive subspaces.

**Recall:** If $X,Y$ are $B$ spaces, a $T \in \mathcal{L}(X,Y)$ is called a compact operator if $\overline{T(B_X)}$ is a compact subset of $Y$.

**Theorem (Pitt):** Let $1 \leq p < q < \infty$, $X = \ell_p$ and $Y = c_0$. Then for every closed subspace $Z$ of $Y$ and every $T \in \mathcal{L}(Z, X)$, $T$ is a compact operator.

**Proof:** Assume $T(B_Z)$ is not relatively compact. Then $\exists \varepsilon > 0$ and $(z_n)_n$ in $B_Z$ s.t. $\forall n \in \mathbb{N}, \|Tz_n - Tz_m\| > \varepsilon$.

Observe that for $n \in \mathbb{N}, \|z_n - z_m\| \geq \frac{\varepsilon}{\|T\|}$.

Denote $(e_i)$; the c.u.b of $X$, $(\hat{e}_i)$; the c.u.b of $X$.

By a diagonal argument, pass to a subseq. $(y_n)_n$ of $(z_n)_n$ s.t. $\forall i \in \mathbb{N}, \lim_{n \to \infty} e_i(y_n) = a_i$ and $\lim_{n \to \infty} e_i^*(Tz_n) = b_i$. 
both exist. Define \( X_0 = Y_n - Y_{n-1} \). Then

\[(X_n)\] is seminormalized \( \left( \frac{\varepsilon}{\|x\|} \leq \inf \|x_n\| , \sup \|x_n\| \leq 2 \right) \)

\& \( \forall \varepsilon > 0 \lim \varepsilon_i (X_n) = 0 \).

Also, \((T_x)\) is seminormalized \( \left( \varepsilon \leq \inf \|T_x\| , \sup \|T_x\| \right) \)

\& \( \forall \varepsilon > 0 \lim \varepsilon_i (T_x) = 0 \).

By passing to subsequences twice, we may assume \((X_n)\) \& \((T_x)\).

Thus, \( \exists C, D > 0 \text{ s.t. } \forall n \in \mathbb{N} \quad \|T_x\| \geq \|x_n\| \geq \|Tx_n\| \geq \frac{1}{D} \|x_n\| \quad \frac{1}{D} \) \n

\( \text{thus } \forall n \in \mathbb{N} \quad \frac{1}{q} - \frac{1}{p} \leq DC\|x\| \) \text{, which is absurd because } 1 \leq q < \infty \).

**Corollary:** If \( X = l^p, 1 \leq p < \infty \), \& \( Y = l^q, 1 < q < \infty \) or \( Y = C_0 \),

then, \( X, Y \) are totally incomparable.

Proof: \( X \hookrightarrow l^p, l \leq p \leq \infty \), \& \( Y \hookrightarrow l^q, \rho < q < \infty \) or \( Y = C_0 \),

then, \( X, Y \) are finitely dimensional.

**Def:** Let \( X, Y \) be B-spaces. A \( \text{TC}(X, Y) \) is called completely continuous if \( X \) is normed \((X, \|\cdot\|_X) \) in \( X, T_x, \|\cdot\|_X \).

**Exercise:** Let \( X, Y \) be B-spaces \& \( \text{TE}(X, Y) \).

i) Show that \( T \) is completely continuous

ii) Under the assumption that \( X \) is reflexive, show that \( T \) is compact if and only if \( H \) is compactly continuous.
Definition: Let $X, Y$ be B-spaces. A T ∈ $L(X, Y)$ is called a strictly singular operator if for every closed infinite dimensional subspace $X_0$ of $X$, $T|_{X_0}: X_0 \to Y$ is not an embedding.

Example: compact operators are strictly singular.

Exercise: Let $X, Y$ be B-spaces & $T \in L(X, Y)$. Then
\[ i) T \text{ is strictly singular} \]
\[ ii) \forall X_0 \subseteq X \text{ inf. dim. } \|T\| > 0 \exists x_0 \in X \text{ s.t. } \|Tx_0\| < \|T\| \|x_0\|. \]

Exercise: Let $X = l_p$, $1 \leq p < \infty$ and $Y = l_q$, $p < q < \infty$ or $Y = c_0$. Then for every $X_0 \subseteq X$ & every $T \in L(X_0, Y)$, $T$ is strictly singular.

Theorem (Pelczynski) Let $X = l_p$, $1 \leq p < \infty$, or $X = c_0$.
Every infinite dimensional complemented subspace of $X$ is isomorphic to $X$.

An $X$ with the above property is called a prime space. (The proof will begin later.)

Def: Let $X, Y$ be B-spaces & $1 \leq p \leq \infty$. Define $X \oplus_p Y = \{ \text{all pairs } (x, y) \in X \times Y \text{ with } \|x\|_p + \|y\|_q \leq \|x\|_p + \|y\|_q \}$
\[ \|x\|_p = \max \{\|x\|_p, 1 \}
\]

Prop: Let $X$ be a B-space s.t. $X = Y \oplus Z$. Then $1 \leq p \leq \infty$
\[ X \cong Y \oplus_p Z. \]

Proof: Let $P: X \to X$ be a projection onto $Y$ with $\text{ker}(P) = Z$ & define $T: X \to Y \oplus_p Z$ by $T_x = (P(x), (I - P)(x))$. Then $\|T\| = \|T\|_p \leq 3 \|P\| \|I - P\|$. \]
Definition: Let $X$ be a $B$-space. We define the $B$-spaces

i) $l^p(X)$, $1 \leq p < \infty$ as all seq. $\mathbf{x} = (x(n))_n$ in $X$

with $||\mathbf{x}||_{l^p(X)} = \left( \sum_{n=1}^{\infty} ||x(n)||^p \right)^{1/p}$ finite.

ii) $c_0(X)$ as all $\mathbf{x} = (x(n))_n$ in $X$ with $\lim_{n \to \infty} ||x(n)|| = 0$

and $||\mathbf{x}||_{c_0(X)} = \max_n ||x(n)||$.

iii) $c_0^\infty(X)$ as all $\mathbf{x} = (x(n))_n$ in $X$ with $\lim_{n \to \infty} ||x(n)|| = \sup_{n \in \mathbb{N}} ||x(n)|| < \infty$.

Remark: $l^p(l^p) \equiv l^p$, for $1 \leq p < \infty$ & $C_0 \equiv C_0(C_0)$.

Take the case $X = l^p$, $1 \leq p < \infty$. We will define an onto linear isometry $T: l^p(l^p) \to l^p(l^p)$. Write $X = \bigcup_{k=1}^{\infty} N_k$, where the $N_k$ are pairwise disjoint and each is infinite.

Write each $N_k$ as $N_k = \{n_1^k < n_2^k < n_3^k < \ldots < n_i^k < \ldots \}$

and for each $x = (x(i))_i \in l^p$ define $\forall k \in \mathbb{N}$, $(T_x)(k)$ $\in l^p$ as $(T_x)(k) = (x(n_i^k))_i$. If $T_x = ((T_x)(k))_k$

we have $||x||^p = \sum_{i=1}^{\infty} |x(i)|^p = \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} x(n_i^k)^p = \sum_{k=1}^{\infty} ||(T_x)(k)||^p = ||Tx||_{l^p(l^p)}^p$. It is also easy to find $\forall \mathbf{x} = (x(n)) \in l^p(l^p)$ an $y \in l^p$ s.t. $Ty = \mathbf{x}$. 