Recall: $C[0,1] = \{ f: [0,1] \to \mathbb{R} \text{ continuous} \}$ with $\| f \|_\infty = \max_{t \in [0,1]} |f(t)|$

**Theorem (Banach-Mazur):** Let $X$ be a separable B-space. Then $X$ embeds isometrically into $C[0,1]$, i.e. $\exists T: X \to C[0,1]$ linear isometry.

**Comment:** The closed subspaces of $C[0,1]$ are all separable Banach spaces.
Preparatory facts:

Def: For a compact Hausdorff top. space $X$, we define $C(X) = \{ f : X \to \mathbb{R} \text{ continuous} \}$. With $||f||_{\infty} = \text{sup}\{ |f(x)| : x \in X \}$ this is a Banach space.

Exercise: Let $X, Y$ be homeomorphic compact Hausdorff spaces. Then $C(X) = C(Y)$ ($\exists T : C(X) \to C(Y)$ cont. isometric).

Proposition: Let $X, Y$ be compact metric spaces (or just compact Hausdorff) and assume that $\exists \phi : X \to Y$ continuous & onto. Then, $\exists T : C(Y) \to C(X)$ isometric embedding.

Proof: For $f \in C(Y)$ we define $Tf \in C(X)$ as follows: $(Tf)(x) = f(\phi(x))$. This is a linear map and $||Tf||_{\infty} = \text{sup}\{ |f(\phi(x))| : x \in X \} = \text{sup}\{ |f(\phi(x))| : x \in X \} = \text{sup}\{ |f(y)| : y \in \phi(X) \} = \text{sup}\{ |f(y)| : y \in Y \} = ||f||_{\infty}$.

Proposition: Let $K$ be a closed subset of $C(0,1)$ (i.e., compact) Then $\exists$ isometric embedding $T : C(K) \to C(0,1)$.

Proof: Take $U = [0,1] \setminus K$ open. Write $U = \bigcup (a_n, b_n)$ where $(a_n, b_n)$ are disjoint open intervals. For $f : K \to K$ cont. define $(Tf)(x) : [0,1] \to [0,1]$ follows: if $t \in K$ then $(Tf)(t) = f(t)$ if $t \notin K$ then $t \in (a_n, b_n)$ for some $n$. Define
Define \( f(t) = \frac{(t-a_n)}{b_n} f(b_n) + \frac{(b_n-t)}{b_n} f(a_n) \)

Then \( \|Tf\| = \|f\| \).

**Proposition:** Let \( X \) be a compact metric space. The \( X \) is homeomorphic to a subset of \( C_0(1)^\mathbb{N} \).

**Proof:** We may assume \( \varnothing \neq \delta d(x, y) \leq 1 \forall \ x, y \in X \). Take a countable dense \( (x_n)_n \in X \) and define \( j : X \to C_0(1)^\mathbb{N} \) by \( j(x) = (d(x, x_n))_n \). Then \( j \) is a continuous bijection, i.e., \( j \) is a homeomorphism.

**Proposition:** \( \exists \phi : \{0, 1\}^\mathbb{N} \to C_0(1)^\mathbb{N} \) onto \& onto.

**Proof:** \( \exists \phi_0 : \{0, 1\}^\mathbb{N} \to C_0(1)^\mathbb{N} \) onto with \( \phi_0(x) = \sum_{n=1}^{\infty} \frac{x(n)}{2^n} \).

Take \( \phi : \{0, 1\}^\mathbb{N} \to \{0, 1\}^\mathbb{N} \) by \( \phi(y) = (\phi_0(y(x)))_n \) which is onto onto. \( 2 : \{0, 1\}^\mathbb{N} \to \{0, 1\}^\mathbb{N} \).

**Proposition:** \( \exists K \subset C_0(1) \), \( K \approx \{0, 1\}^\mathbb{N} \)

**Proof:** define \( \phi : \{0, 1\}^\mathbb{N} \to [0, 1] \) with

\[
\phi(x) = \sum_{n=1}^{\infty} \frac{2x(n)}{3^n}
\]
Proposition: Let \( X \) be a compact metric space. Then \( C(X) \) embeds isometrically into \( C_{0,1} \).

Proof: We may assume \( X \) is a closed subset of \( [0,1] \). Take \( \Phi: [0,1] \to [0,1] \) onto and define

\[
Y = \Phi'(X), \quad \text{closed (compact) subset of } [0,1],
\]

which we may identify with a closed subset of \( [0,1] \). Because \( \Phi: Y \to X \) onto \( \Rightarrow \ C(X) \xrightarrow{\text{iso}} C(Y) \).

But \( C(Y) \xrightarrow{\text{iso}} C_{0,1} \).

Proposition: Let \( X \) be a separable B-space. Then \( X \) embeds isometrically into \( C_{0,1} \).

Proof: it suffices to find a compact metric space \( Y \) s.t. \( X \xrightarrow{\text{iso}} C(Y) \).

Let \( Y = (B_{X^+}, w^+) \) which is compact.

Remark: \( \forall x \in X, \ x: (X^+, w^+) \to \mathbb{R} \) linear form.

\[
(w^+ \text{ is the smallest sup norm on } X^+ \text{ making all } x \text{ cont.})
\]

\[
\Rightarrow \ X \xrightarrow{\text{iso}} B_{X^+} \subseteq C(Y) \text{ and }
\]

\[
\|x\|_{w^+} = \sup \|x^*(x)\|_{X^*} = \|x\|_{B_{X^+}} \|w^+\|_{\infty}. \text{ Then, } T: X \to C(Y)
\]

given by \( Tx = \|x\|_{B_{X^+}} \) is a linear isometry. Remains to recall that when \( X \) is separable, \( (B_{X^+}, w^+) \)
is metrizable. To see this, take \( X^0 \) dense in \( B_X \) and define \( T \) on \( X^* \) to be the smallest topology making all \( x_{i,n} \in X^0 \) continuous. Then \( T \) is metrizable \& \( T \subset w^* \) 

\( B_{X^1} \) (\( B_{X^1} \)) is compact and \( id : (B_{X^1}) - (B_{X^1}) \) is a continuous bijection, i.e., it is a homeomorphism.

**Exercise:** Let \( (X_i, T) \) be a compact Hausdorff space. Define \( \delta : X \rightarrow (C(X))^* \) as follows:

\[(d(x))(f) = f(x)\]

i) Show that \( \delta \) is well defined, i.e., \( \delta(x) \in (C(X))^* \) \( \forall x \in X \) and that \( ||\delta(x)|| = 1 \) \( \forall x \in X \).

ii) Show that \( \delta : (X_i, T) \rightarrow ((C(X))^*, w^*) \) is continuous \& 1-1.

iii) Deduce that \( (X_i, T) \) is metrizable if and only if \( C(X) \) is separable.