Problem 1. Let $X$ be a Banach space. Let $(x_n)_n$ be a bounded sequence in $X$ that has no norm convergent subsequence. Prove that there exists a subsequence $(y_n)_n$ of $(x_n)_n$ so that if for all $n$ we set $z_n = y_{2n} - y_{2n-1}$ then $(z_n)_n$ is Schauder basic.

Problem 2. Let $(X, \tau)$ be a compact Hausdorff topological space. Prove that $(X, \tau)$ is metrizable if and only if $C(X)$ is separable.

Problem 3. Let $X$ be a Banach space and let $(x_n)_n$ be a bounded sequence in $X$. Prove that all following statements are equivalent.

(i) The sequence $(x_n)_n$ is Schauder basic and there exists $x^*_0 \in X^*$ with $x^*_0(x_n) = 1$ for all $n \in \mathbb{N}$.

(ii) The sequence $(x_n)_n$ is Schauder basic and there exists $b > 0$ so that for all $a_1, \ldots, a_n \in \mathbb{R}$ we have

$$\left\| \sum_{i=1}^n a_i x_i \right\| \geq b \left| \sum_{i=1}^n a_i \right|.$$

(iii) The sequence $(x_n)_n$ is Schauder basic and there exists $c > 0$ so that for all $a_1, \ldots, a_n \in \mathbb{R}$ we have

$$\left\| \sum_{i=1}^n a_i x_i \right\| \geq c \left\| \sum_{i=1}^n a_i s_i \right\|,$$

where $(s_i)_i$ denotes the summing basis of $c_0$.

(iv) The sequence $(d_n)_n$ given by $d_1 = x_1$ and $d_n = x_n - x_{n-1}$, for $n \geq 2$, is Schauder basic.
Problem 1: Set $S = \{x_n : n \in \mathbb{N}\}$ & distinguish two cases:

i) $S$ is not relatively w-compact. In this case, since $S$ is bounded, it has a subsequence $(x_{n_k})_k$ of $(x_n)n$ that is already S-basic. Then, $(2n)_n$ given by $2n = y_{2n} - y_{2n-1}$, $n \in \mathbb{N}$, is S-basic as a block of an S-basic sequence.

ii) $S$ is relatively w-compact. By Eberlein-Šmulian, there exists $(x_{n_k})_k$ & $x_0 \in X$ s.t. $x_{n_k} \rightarrow w x_0$. But $(x_{n_k})$ is not Cauchy, thus we may find a further subsequence $2n \in \mathbb{N}$ s.t. $\|x_{2n_k} - x_{2n_k-1}\| > \eta_k \forall k$ & of course, $x_{2n_k} - x_{2n_k-1} \rightarrow w D$. Thus, it has an S-basic subsequence $(x_{2n_{jk}} - x_{2n_{jk}-1})$.

Define $y_{2j} = x_{2n_{jk}}$, $y_{2j-1} = x_{2n_{jk}-1}$.

Problem 2: Steps outlined in lecture 11.

Problem 3: We will prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). We will also as an example, (i) $\Rightarrow$ (iv). The other direction (iv) $\Rightarrow$ (i) is conceptually similar. Let $m \in \mathbb{N}$ & $a_1, \ldots, a_n \in \mathbb{R}$. 
(i) = (ii) \( b = \frac{1}{\|x_0^*\|} \) \( x_0 \) work. (ii) = (i) Define \( x_0^* \) on \( \langle x_n : n \geq 1 \rangle \), where \( b \geq \|x_0^*\| \) \( b \). Extend to \( \langle x_n : n \geq 1 \rangle \) and then to \( X \) by \( b \).

(iii) = (ii) is obvious. (ii) = (iii) If \( (x_n) \) is \( K \)-basic then for \( m < n \)

\[ \|\sum_{i=1}^{n} a_i x_i\| = \frac{1}{1-K} \|\sum_{i=1}^{m} a_i x_i\| \geq \frac{b}{1-K} \max_{i=m+1}^{n} |a_i|, \]

\[ \|\sum_{i=1}^{n} a_i x_i\| \geq \frac{b}{1-K} \max_{i=m}^{n} |a_i| = \frac{b}{1-K} \|\sum_{i=1}^{n} a_i x_i\|. \]

(iv) Assume \( (x_n) \) is \( K \)-basic, \( \|x_n^*\| = C, \sup \|x_n\| = D. \)

Also, set \( a_m = 0 \) & denote \( y = \sum_{i=1}^{n} a_i x_i. \)

Then,

\[ \|\sum_{i=1}^{m-1} a_i d_i\| = \|\sum_{i=1}^{m-1} (a_i - a_{i+1}) x_i + a_m x_m\| \]

\[ \leq K \|\sum_{i=1}^{m-1} (a_i - a_{i+1}) x_i + a_m x_m - \sum_{i=m+1}^{n} (a_i - a_{i+1}) x_i\| \]

\[ = K \|\sum_{i=1}^{m-1} (a_i - a_{i+1}) x_i + (a_m - a_{m+1}) x_m + \sum_{i=m+1}^{n} (a_i - a_{i+1}) x_i - a_m x_m\| \]

\[ \leq K \|\sum_{i=1}^{n} (a_i - a_{i+1}) x_i\| + K |a_m - a_{m+1}| \|x_m\| \]

\[ = k \|\sum_{i=1}^{n} a_i d_i\| + K \|\sum_{i=1}^{n} (a_i - a_{i+1}) - a_m - a_{m+1}\| \|x_m\| \]

\[ \leq k \|y\| + 2KD \left[ \|x_0^* (\sum_{i=1}^{m} (a_i - a_{i+1}) x_i)\| + \|x_0^* (\sum_{i=m}^{n} a_i x_i)\| \right] \]

\[ \leq k \|y\| + 2KD \left[ \|x_0^* \| \|\sum_{i=1}^{m} (a_i - a_{i+1}) x_i\| + \|x_0^*\| \|y\| \right] \]

\[ \leq k \|y\| + 2KD \left[ \|x_0^*\| \|\sum_{i=1}^{m} (a_i - a_{i+1}) x_i\| + \|x_0^*\| \|y\| \right] \]

\[ = K (1 + 2 KD \|x_0^*\| + 2D \|x_0^*\|) \|y\| \]