Stochastic Model Efficiency Applications: Cluster-Distance Sampling and Parametric Curve Fitting to Tackle Sampling Errors and Bias

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Practitioners/researchers are challenged to make credible inferential statements about the population distribution of important economic variables.

These distributions often involve a large number of policyholders and economic scenarios.

Well known challenge of running a stochastic asset/liability model is the long run-time.

Successful projection of these population distributions is important for actuaries:
  - Pricing, reserving, budgeting risk capital.
To analyze the population distribution of economic outcomes, model efficiency approaches are often utilized.

**Model Efficiency**: Mathematical approaches that reduce the number of economic scenarios required to achieve a given level of precision in stochastic actuarial modeling (Rosner 2011).

Model efficiency approaches include (Rosner 2011):

- Transfer Scenario Order
- Importance Sampling
- Curve Fitting
- **Representative Scenarios**
The CWU Computer Science department engineered these two packages to implement our research and empower future model efficiency research:

- **CSTEP**: Cluster Sampling for Tail Estimation of Probability (reduce sampling errors, especially at the tail of a distribution)
- **AMOOF2**: Actuarial Model Optimal Outcome Fit V2.0 (reduce sampling bias)

We use CSTEP and AMOOF2 to analyze statutory ending surplus data from a real block of variable annuities, provided by Milliman (Milliman data).

Upgrade from SALMS (Stochastic Asset Liability Model Sampling) used since 2003

CSTEP is open source, high performance computation software
- Universe capacity: 8,388,608 scenarios with up to 4500 time periods each
- Flexible sample size, reversible and reusable sampling
- Rate sampling (interest rate, equity return, index)
Consider a population of $N$ rate paths

Editable distance formulas similar to Euclidean distance are used to select $n$ representative (pivot) scenarios where $n \ll N$ (Chueh 2002)

- Choose an arbitrary path out of the $N$ paths and call it Pivot 1
- Calculate the distances of the remaining $N - 1$ paths to Pivot 1, the path with the largest distance to Pivot 1 is Pivot 2
- Assign each of remaining $N - 2$ paths to the closest of Pivots 1 and 2, forming two disjoint paths
- Calculate the distances of the remaining $N - 2$ paths to Pivots 1 and 2, the path with the largest distance to Pivots 1 and 2 is Pivot 3
- Assign each of remaining $N - 3$ paths to the closest of Pivots 1, 2, and 3, forming three disjoint paths ... repeat until $n$ Pivots

A probability is then assigned to each representative scenario
In order to employ the method of representative scenarios, we need to be able to calculate the distance between two scenarios and tie that distance to the model output.

The original version of CSTEP employed a theorem of high-dimensional continuity (Continuity Theorem 1).

The new version of CSTEP employs a modified theorem of high-dimensional continuity that improves the tail fit for volative economic scenarios and equity-based insurance guarantees (Continuity Theorem 2).
Consider two n-period rate scenarios:

\[ x = (r_1, r_2, ..., r_n) \] and \[ s = (r'_1, r'_2, ..., r'_n) \]

Let \( CF_t \) denote the net cash flow at the end of period \( t \) under scenario \( x \); \( CF'_t \) is similarly defined under scenario \( s \).

Let \( f(x) = \sum_{t=1}^{n} CF_t \prod_{k=1}^{t} \frac{1}{1+r_k} \) and \( f(s) = \sum_{t=1}^{n} CF'_t \prod_{k=1}^{t} \frac{1}{1+r'_k} \)

Let the distance between \( x \) and \( s \):

\[ d_X(x, s) = \sqrt{\sum_{t=1}^{n} [\prod_{k=1}^{t} \frac{1}{1+r_k} - \prod_{k=1}^{t} \frac{1}{1+r'_k}]^2} \]

Let the distance between \( f(x) \) and \( f(s) \):

\[ d_Y(f(x), f(s)) = | \sum_{t=1}^{n} [CF_t \prod_{k=1}^{t} \frac{1}{1+r_k} - CF'_t \prod_{k=1}^{t} \frac{1}{1+r'_k}] | \]
Given an arbitrary risk scenario \( s \in X, \forall \epsilon > 0 \)

\[ \exists \ delta = \frac{\epsilon}{2 \sqrt{nM}} \]

\[ \exists \forall \ scenario \ vectors \ x \in X: \]

\[ d_X(x, s) \leq \delta \ is \ a \ sufficient \ condition \ \exists \ d_Y(f(x), f(s)) \leq \epsilon \]

\[ M = max_t(|CF_t|, |CF'_t|) \]

The above illustrates uniform continuity
Continuity Theorem 1: Proof

\[ [d_Y(f(x), f(s))]^2 = | \sum_{t=1}^{n} [CF_t \prod_{k=1}^{t} \frac{1}{1+r_k} - CF'_t \prod_{k=1}^{t} \frac{1}{1+r'_k}]^2 \]
\[ \leq (2M)^2 | \sum_{t=1}^{n} [\prod_{k=1}^{t} \frac{1}{1+r_k} - \prod_{k=1}^{t} \frac{1}{1+r'_k}]^2 \]
\[ \leq n(2M)^2 \sum_{t=1}^{n} [\prod_{k=1}^{t} \frac{1}{1+r_k} - \prod_{k=1}^{t} \frac{1}{1+r'_k}]^2 \]
\[ = n(2M)^2[d_X(x, s)]^2 \]
\[ \leq n(2M)^2 \delta^2 \]
\[ = \epsilon^2 \]

Then: \( d_X(x, s) \leq \delta \) ensures that \( d_Y(f(x), f(s)) \leq \epsilon \)
Original CSTEP Distance Formulas

Let \( p \) denote a pivot scenario

**Significance Method:**
\[
d_X(x, p) = \sqrt{\sum_{t=1}^{n} \left[ \prod_{k=1}^{t} \frac{1}{1+r_k} \right]^2}
\]

**Relative Present Value (RPV) Method:**
\[
d_X(x, p) = \sqrt{\sum_{t=1}^{n} \left[ \prod_{k=1}^{t} \frac{1}{1+r_k} - \prod_{k=1}^{t} \frac{1}{1+r_p} \right]^2}
\]
Continuity Theorem 2

- Let $C_t \approx CF_t$ and $C'_t \approx CF'_t$ (determined by historical experience for a similar block of business or a sample(s) of the population distribution)

- Let the distance between $x$ and $s$:

$$d^*_X(x, s) = \sqrt{\sum_{t=1}^{n}[C_t \prod_{k=1}^{t} \frac{1}{1+r_k} - C'_t \prod_{k=1}^{t} \frac{1}{1+r'_k}]^2}$$

- Given an arbitrary risk scenario $s \in X$, $\forall \epsilon > 0$

$$\exists \delta = \epsilon > 0$$

$$\exists \forall \text{ scenario vectors } x \in X:$$

$$d^*_X(x, s) \leq \delta \text{ is a sufficient condition } \exists \ dy(f(x), f(s)) \leq \epsilon$$
Continuity Theorem 2: Proof

\[ [d_Y(f(x), f(s))]^2 = | \sum_{t=1}^{n} [CF_t \prod_{k=1}^{t} \frac{1}{1+r_k} - CF'_t \prod_{k=1}^{t} \frac{1}{1+r'_k}] |^2 \]

\[ \approx | \sum_{t=1}^{n} [C_t \prod_{k=1}^{t} \frac{1}{1+r_k} - C'_t \prod_{k=1}^{t} \frac{1}{1+r'_k}] |^2 \]

\[ = [d^*_X(x, s)]^2 \]

\[ \leq \delta^2 \]

\[ = \epsilon^2 \]

Then: \( d^*_X(x, s) \leq \delta \) ensures that \( d_Y(f(x), f(s)) \leq \epsilon \)
Let $p$ denote a pivot scenario

**Economic Significance Method:**

$$d^*_X(x, p) = \sqrt{\sum_{t=1}^{n} [C_t \prod_{k=1}^{t} \frac{1}{1+r_k}]^2}$$

**Economic Present Value (EPV) Method:**

$$d^*_X(x, p) = \sqrt{\sum_{t=1}^{n} [C_t \prod_{k=1}^{t} \frac{1}{1+r_k} - C'_t \prod_{k=1}^{t} \frac{1}{1+r'_k}]^2}$$
AMOOF2: Actuarial Model Optimal Outcome Fit, Version 2.0 (Chueh and Curtis 2004)

Stand alone desktop software suite communicating to Microsoft Excel and incorporating formulas computed by Wolfram Mathematica 8.0

Open source, high computation software for complex probability distribution fitting for stochastic modeling (principle based approach, actuarial guideline 43)
AMOOF2 (Continued)

- Allows for efficient determination of a data set’s summary statistics (such as mean and variance) and tail metrics (such as VaR and CTE)
- Implements pdf selection (both 22 single and mixed distributions, Klugman (2008)), graphing features to aid user flexibility, and high-computation outcome reporting
- Implements small bias adjustments arising from maximum likelihood estimation (MLE)
Fitting probability density functions
- MLE: Analytic MLEs for 22 distributions in closed form completed using Mathematica 8.0 (if exist), otherwise, Excel’s solver is used to determine MLEs
- Methods of Moments: First four positive and negative theoretical moments can be set equal to their corresponding sample moments

Small Sample Bias-Corrected MLEs (BMLEs)
- Cox and Snell/Cordeiro and Klein (CSCK) analytic BMLEs for 15/22 distributions; remaining distributions do not have closed form BMLEs (Cox and Snell 1968, Cordeiro and Klein 1994)

Integration: VaR and CTE
- High-Precision Reimann Sums (Gaussian Quadrature Integration)
Consider a distribution with $p$ parameters: $\theta = (\theta_1, \theta_2, \ldots, \theta_p)'$

Define joint cumulants based on the total loglikelihood function $l(\theta)$ with $n$ observations for $i, j, l = 1, 2, \ldots, p$:

$$\kappa_{ij} = E\left[\frac{\partial^2 l}{\partial \theta_i \partial \theta_j}\right]$$

$$\kappa_{ijl} = E\left[\frac{\partial^3 l}{\partial \theta_i \partial \theta_j \partial \theta_l}\right]$$

Cumulant derivative: $\kappa_{ij}^{(l)} = \frac{\partial \kappa_{ij}}{\partial \theta_l}$

Total Fisher information Matrix of order $p$ for $\theta$ is $K = \{-\kappa_{ij}\}$; inverse is $K^{-1} = \{-\kappa_{ij}\}$
CSCK showed if all cumulants are assumed to be $O(n)$:

$$b_s = E[\hat{\theta}_s - \theta_s] = \sum_{i=1}^{p} \kappa^s_i \sum_{j,l=1}^{p} [\kappa^{(l)}_{ij} - 0.5 \kappa_{ijl}] k^{il} + O(n^{-2})$$

Bias vector $b = E[\hat{\theta} - \theta] = K^{-1} \text{Avec}[K^{-1}] + O(n^{-2})$

where $A = \{ A^{(1)} | A^{(2)} | \ldots | A^{(p)} \}$ and $A^{(1)} = \{ \kappa^{(l)}_{ij} - 0.5 \kappa_{ijl} \}$ for $l = 1, 2, \ldots, p$

The BMLE vector ($\tilde{\theta}$) is the difference between the MLE vector and the MLE bias vector evaluated at the MLEs:

$$\tilde{\theta} = \hat{\theta} - \hat{b}$$
b[f_, p_] := Module[{l, Gradient, Hessian, ThirdPartialDer, ExpectHessian, ExpectThirdPartialDer, DerivativeExpectHessian, aijk, Amatrix, Kinv, vecKinv, BIAS, Expect},

Expect[x_] := Integrate[x*f, {yi, 0, ∞}, Assumptions -> {θ1 ∈ Reals, θ2 ∈ Reals, θ3 ∈ Reals, θ1 > 0, θ2 > 0, θ3 > 0}];

SuperLog[On];

l = Log[Π_{i=1}^{n}f];
Gradient = D[l, {p}];
Hessian = D[l, {p, 2}];
ThirdPartialDer = D[l, {p, 3}];
ExpectHessian = Map[Expect[#] &, Hessian];
ExpectThirdPartialDer = Map[Expect[#] &, ThirdPartialDer];
DerivativeExpectHessian = D[ExpectHessian, {p}];
aijk = DerivativeExpectHessian - ExpectThirdPartialDer/2;
Amatrix = Apply[Join, aijk~Join~{2}];
Kinv = Inverse[-ExpectHessian]; vecKinv = Flatten[Transpose[Kinv]]; 
BIAS = Simplify[Kinv.Amatrix.vecKinv];

SuperLog[Off]; BIAS]
Milliman Data

- Present value of ending surplus data at 30 years (360 months) was the stochastic model output from a real block of variable annuities using a proprietary stochastic scenario generator.

- Asset and liability cash flows are on a monthly basis.

- Inforce distribution is allocated among six funds: a general cash fund and five other funds (bond and equity mutual funds); we assume the portfolio is rebalanced each period.

- Monthly portfolio yield rates were obtained from the 7-year US treasury yield and five stock index returns via the asset allocation rule.
10,000 stochastic economic interest rate scenarios were considered, where each scenario is a random path of monthly portfolio yield rates $x = (r_1, r_2, \ldots, r_{360})$

The tax rate is zero: business is offshore in the model with no taxes

We call the 10,000 real model outcome data the “full-run distribution”
Milliman currently uses a compression process called **cluster sampling** to efficiently model millions of policies into a smaller number of model points (Freedman and Reynolds 2008)

Cluster sampling automatically assigns all policies from a seriatum inforce file to one of a small user-selected number of representative model points.

The Euclidian distance between policies is calculated by comparing location variables, and each policy’s “importance” is determined as the product of its size (such as face amount or account value) and distance from its nearest neighboring policy.

Cluster sampling assigns the least important policy to its neighbor and grosses up the inforce amount for that neighbor.

The process is repeated iteratively until the model is a size specified by the user.
We want to compare our representative scenarios approach to Milliman’s cluster sampling approach.

Specifically, we want to determine which method produces a sampling distribution that best replicates summary statistics and tail metrics from the full-run distribution.
First, we compare cluster sampling to representative scenarios (significance method and RPV method): each sample will consist of 50 scenarios

- Compare summary statistics between all methods (mean, median, standard deviation, minimum, maximum)
- Compare worst present value of ending surplus CTE between all methods
- Compare worst present value of ending surplus CTE for different nestings of RPV method
- Determine effect of scenario size on worst present value of ending surplus CTE for RPV method

CSTEP is used to obtain the representative scenarios (significance method and RPV method), and AMOOF2 is used to determine summary statistics and CTE
# Summary Statistics (Percentage of Full Run)

<table>
<thead>
<tr>
<th></th>
<th>FULL RUN: 10,000 SCENARIOS</th>
<th>CLUSTER SAMPLING: 50 SCENARIOS</th>
<th>SIGNIFICANCE METHOD: 50 SCENARIOS</th>
<th>RPV METHOD: 50 SCENARIOS</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Mean:</strong></td>
<td>7,771,679</td>
<td>7,528,719 (96.87%)</td>
<td>7,776,994 (100.07%)</td>
<td>14,220,751 (182.98%)</td>
</tr>
<tr>
<td><strong>Median:</strong></td>
<td>7,454,651</td>
<td>7,324,723 (98.25%)</td>
<td>7,151,056 (95.93%)</td>
<td>18,458,682 (247.62%)</td>
</tr>
<tr>
<td><strong>Standard Deviation:</strong></td>
<td>8,352,868</td>
<td>8,250,754 (98.78%)</td>
<td>8,562,364 (102.51%)</td>
<td>9,191,080 (110.04%)</td>
</tr>
<tr>
<td><strong>Minimum:</strong></td>
<td>-23,450,779</td>
<td>-8,002,320 (34.12%)</td>
<td>-14,858,570 (63.36%)</td>
<td>-22,143,294 (94.42%)</td>
</tr>
<tr>
<td><strong>Maximum:</strong></td>
<td>53,225,896</td>
<td>33,895,093 (63.68%)</td>
<td>26,103,344 (49.04%)</td>
<td>18,458,682 (34.68%)</td>
</tr>
</tbody>
</table>
Worst Present Value of Ending Surplus CTE by All Non-Economic Methods, 50 Scenarios (Percentage of Full Run)

<table>
<thead>
<tr>
<th>CTE LEVEL</th>
<th>FULL RUN: 10,000 SCN</th>
<th>CLUSTER SAMPLING: 50 SCN</th>
<th>SIGNIFICANCE METHOD: 50 SCN</th>
<th>RPV METHOD: 50 SCN</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%</td>
<td>7,771,679</td>
<td>7,528,719 (96.87%)</td>
<td>7,776,994 (100.07%)</td>
<td>14,220,751 (182.98%)</td>
</tr>
<tr>
<td>50%</td>
<td>1,423,770</td>
<td>1,057,077 (74.24%)</td>
<td>994,639 (69.86%)</td>
<td>9,982,821 (701.15%)</td>
</tr>
<tr>
<td>70%</td>
<td>-1,393,692</td>
<td>-1,899,284 (136.28%)</td>
<td>-1,986,828 (142.56%)</td>
<td>4,332,248 (-310.85%)</td>
</tr>
<tr>
<td>90%</td>
<td>-6,499,169</td>
<td>-5,710,820 (87.87%)</td>
<td>-7,272,823 (111.90%)</td>
<td>-8,224,232 (126.54%)</td>
</tr>
<tr>
<td>95%</td>
<td>-9,283,170</td>
<td>-7,372,458 (79.42%)</td>
<td>-10,793,427 (116.27%)</td>
<td>-13,945,534 (150.22%)</td>
</tr>
<tr>
<td>98%</td>
<td>-12,704,701</td>
<td>-8,002,320 (62.99%)</td>
<td>-14,858,570 (116.95%)</td>
<td>-15,093,811 (118.80%)</td>
</tr>
<tr>
<td>99%</td>
<td>-15,141,284</td>
<td>-8,002,320 (52.85%)</td>
<td>-14,858,570 (98.13%)</td>
<td>-16,190,443 (106.93%)</td>
</tr>
</tbody>
</table>
### Worst Present Value of Ending Surplus CTE by Nested RPV, 50 Scenarios (Percentage of Full Run)

<table>
<thead>
<tr>
<th>CTE LEVEL</th>
<th>FULL RUN: 10,000 SCN</th>
<th>RPV METHOD: 50 SCN</th>
<th>RPV METHOD (2 NESTED): 50 SCN</th>
<th>RPV METHOD (3 NESTED): 50 SCN</th>
<th>RPV METHOD (5 NESTED): 50 SCN</th>
<th>RPV METHOD (10 NESTED): 50 SCN</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%</td>
<td>7,771,679</td>
<td>14,220,751 (182.98%)</td>
<td>18,401,442 (236.78%)</td>
<td>8,546,740 (109.97%)</td>
<td>12,307,705 (158.37%)</td>
<td>7,041,167 (90.60%)</td>
</tr>
<tr>
<td>50%</td>
<td>1,423,770</td>
<td>9,982,821 (701.15%)</td>
<td>4,458,664 (313.16%)</td>
<td>3,934,156 (276.32%)</td>
<td>2,086,664 (146.56%)</td>
<td>1,413,245 (99.26%)</td>
</tr>
<tr>
<td>70%</td>
<td>-1,393,692</td>
<td>4,332,248 (-310.85%)</td>
<td>-2,518,380 (180.70%)</td>
<td>1,242,003 (-89.12%)</td>
<td>-1,419,559 (101.86%)</td>
<td>-1,062,040 (76.20%)</td>
</tr>
<tr>
<td>90%</td>
<td>-6,499,169</td>
<td>-8,224,232 (126.54%)</td>
<td>-8,429,374 (129.70%)</td>
<td>-7,574,795 (116.55%)</td>
<td>-9,715,185 (149.48%)</td>
<td>-6,235,984 (95.95%)</td>
</tr>
<tr>
<td>95%</td>
<td>-9,283,170</td>
<td>-13,945,534 (150.22%)</td>
<td>-13,332,423 (143.62%)</td>
<td>-11,012,146 (118.62%)</td>
<td>-11,684,922 (125.87%)</td>
<td>-9,245,241 (99.59%)</td>
</tr>
<tr>
<td>98%</td>
<td>-12,704,701</td>
<td>-15,093,811 (118.80%)</td>
<td>-14,784,058 (116.37%)</td>
<td>-15,938,785 (125.46%)</td>
<td>-14,185,185 (111.65%)</td>
<td>-16,975,119 (133.61%)</td>
</tr>
<tr>
<td>99%</td>
<td>-15,141,284</td>
<td>-16,190,443 (106.93%)</td>
<td>-16,611,259 (109.71%)</td>
<td>-16,752,512 (110.64%)</td>
<td>-17,552,041 (115.92%)</td>
<td>-19,664,146 (129.87%)</td>
</tr>
</tbody>
</table>
## Worst Present Value of Ending Surplus CTE by Nested RPV, 100 Scenarios (Percentage of Full Run)

<table>
<thead>
<tr>
<th>CTE LEVEL</th>
<th>FULL RUN: 10,000 SCN</th>
<th>RPV METHOD: 100 SCN</th>
<th>RPV METHOD (2 NESTED): 100 SCN</th>
<th>RPV METHOD (3 NESTED): 100 SCN</th>
<th>RPV METHOD (5 NESTED): 100 SCN</th>
<th>RPV METHOD (10 NESTED): 100 SCN</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%</td>
<td>7,771,679</td>
<td>15,829,929 (203.69%)</td>
<td>12,269,702 (157.88%)</td>
<td>6,894,332 (88.71%)</td>
<td>6,956,984 (89.52%)</td>
<td>12,064,948 (155.24%)</td>
</tr>
<tr>
<td>50%</td>
<td>1,423,770</td>
<td>49,318 (3.46%)</td>
<td>1,833,503 (128.78%)</td>
<td>3,257,421 (228.79%)</td>
<td>2,741,708 (192.57%)</td>
<td>3,766,478 (264.54%)</td>
</tr>
<tr>
<td>70%</td>
<td>-1,393,692</td>
<td>-1,610,698 (115.57%)</td>
<td>-785,821 (56.38%)</td>
<td>455,545 (-32.69%)</td>
<td>-558,315 (40.06%)</td>
<td>393,763 (-28.25%)</td>
</tr>
<tr>
<td>90%</td>
<td>-6,499,169</td>
<td>-8,629,475 (132.78%)</td>
<td>-8,994,109 (138.39%)</td>
<td>-8,772,441 (134.98%)</td>
<td>-7,264,783 (111.78%)</td>
<td>-7,084,628 (109.01%)</td>
</tr>
<tr>
<td>95%</td>
<td>-9,283,170</td>
<td>-12,767,071 (137.53%)</td>
<td>-11,033,022 (118.85%)</td>
<td>-11,782,974 (126.93%)</td>
<td>-11,860,918 (127.77%)</td>
<td>-11,117,300 (119.76%)</td>
</tr>
<tr>
<td>98%</td>
<td>-12,704,701</td>
<td>-13,929,242 (109.64%)</td>
<td>-14,472,082 (113.91%)</td>
<td>-14,547,520 (114.51%)</td>
<td>-15,075,262 (118.66%)</td>
<td>-16,245,105 (127.87%)</td>
</tr>
<tr>
<td>99%</td>
<td>-15,141,284</td>
<td>-15,576,209 (102.87%)</td>
<td>-16,574,241 (109.46%)</td>
<td>-16,261,578 (107.40%)</td>
<td>-17,109,332 (113.00%)</td>
<td>-17,459,031 (115.31%)</td>
</tr>
</tbody>
</table>
Milliman Analysis I: Observations

- With 50 scenarios, cluster sampling provided a sample with summary statistics that best matched the full-run distribution.

- With 50 scenarios, the significance method provided a sample with CTEs that best matched the full-run distribution (except for CTE70).

- Increasing scenario size from 50 to 100 did substantially improve the CTE of the sample-run distribution for the RPV method.

- With 50 scenarios and using just the RPV method, a higher nesting of scenarios tended to produce a total sample with the best CTEs; this was reversed for 100 scenarios.
Second, we compare non-economic methods to economic methods: each sample will consist of 50 scenarios

- Non-economic methods: Significance and RPV
- Economic methods: Economic Significance and EPV

To obtain $C_t$, the RPV method was used to select a sample of 100 scenarios; $C_t$ was determined as the average net cash flow at each time
Worst Present Value of Ending Surplus CTE by All Methods, 50 Scenarios (Percentage of Full Run)

<table>
<thead>
<tr>
<th>CTE LEVEL</th>
<th>FULL RUN: 10,000 SCN</th>
<th>SIGNIFICANCE METHOD: 50 SCN</th>
<th>ECONOMIC SIGNIFICANCE METHOD: 50 SCN</th>
<th>RPV METHOD: 50 SCN</th>
<th>EPV METHOD: 50 SCN</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%</td>
<td>7,771,679</td>
<td>7,776,994 (100.07%)</td>
<td>7,771,676 (100.00%)</td>
<td>14,220,751</td>
<td>9,001,950 (115.83%)</td>
</tr>
<tr>
<td>50%</td>
<td>1,423,770</td>
<td>994,639 (69.86%)</td>
<td>1,211,631 (85.10%)</td>
<td>9,982,821</td>
<td>4,245,893 (298.21%)</td>
</tr>
<tr>
<td>70%</td>
<td>-1,393,692</td>
<td>-1,986,828 (142.56%)</td>
<td>-1,642,666 (117.86%)</td>
<td>4,332,248</td>
<td>-1,078,845 (-77.41%)</td>
</tr>
<tr>
<td>90%</td>
<td>-6,499,169</td>
<td>-7,272,823 (111.90%)</td>
<td>-8,465,724 (130.26%)</td>
<td>-8,224,232</td>
<td>-6,312,480 (97.13%)</td>
</tr>
<tr>
<td>95%</td>
<td>-9,283,170</td>
<td>-10,793,427 (116.27%)</td>
<td>-11,578,558 (124.73%)</td>
<td>-13,945,534</td>
<td>-9,833,901 (105.93%)</td>
</tr>
<tr>
<td>98%</td>
<td>-12,704,701</td>
<td>-14,858,570 (116.95%)</td>
<td>-15,355,025 (120.86%)</td>
<td>-15,093,811</td>
<td>-12,969,910 (102.09%)</td>
</tr>
<tr>
<td>99%</td>
<td>-15,141,284</td>
<td>-14,858,570 (98.13%)</td>
<td>-15,355,025 (101.41%)</td>
<td>-16,190,443</td>
<td>-15,534,818 (102.60%)</td>
</tr>
</tbody>
</table>
Economic methods (which considered net cash flow at each time) tended to outperform non-economic methods in terms of measuring CTE.

- Economic significance method provided better CTE results at lower levels, whereas EPV method provided better CTE results at higher levels.
Conclusions

- We have implemented a set of sampling techniques in CSTEP to enhance the precision of tail metrics aimed within a compressed run time allowance.

- We have also developed AMOOF2 which provides a nice graphical user interface platform with 22 visual probability density models which can fit single pdfs or mixed pdfs estimated using maximum likelihood estimation.

- We hope we have established a precedent as developing open-source software for research and education will benefit the industry and all stochastic modelers in tackling sampling bias and error that are critical to model efficiency and quality of risk managing, reporting, and model refining.
Next Steps

- Consider scenario sampling using the economic significance method and EPV method
  - Major issue: How to best determine $C_t$?

- Implement curve fitting analyses using AMOOF2
  - Fit various single and mixed probability density functions to model outcome data
  - Refine CSCK method so that calculations are more efficient and can be applied to distributions with a high number of parameters (perhaps use different programming language?)

- Other sensitivity testing
  - Vary duration of rates in scenarios, conduct analyses in Rosner (2011)


