3. Plane Partitions and the ASM-DPP

Conjecture

DPP = Descending Plane Partitions (see def below)

Conjecture (Mills-Robbins-Rumsey '82)
the numbers of ASM (n x n) and DPP (order n)
are the same, including some refinements.
OBSERVABLES FOR ASMs

- \#(-1) number of -1's
- inversion number

\[ \text{Inv}(A) = \sum_{i<j} \sum_{k<l} A_{il} A_{jk} \]

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
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\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}

\#(f1) 0 0 0 0 0 0 0 1

Inv 0 1 1 2 2 3 3 2

Inv - #(f1) 0 1 1 2 2 3 3 1
6 Vertex

by symmetry:
\[
\begin{align*}
N_{a_1} &= N_{a_2} = \frac{N_a}{2} \\
N_{b_1} &= N_{b_2} = \frac{N_b}{2} \\
N_{c_1} &= N_{c_2} + n \\
N_c &= N_{c_1} + N_{c_2}
\end{align*}
\]

\[\#(-1) = N_{c_2} = \frac{N_c - n}{2}\]

\[\text{Inv} = N_{a_1} + N_{c_2}\]

\[\text{Inv} - \#(-1) = N_{a_1} = \frac{N_a}{2}\]
Recall $Z_{6V} (a, b, c) = \sum_{6V\text{configs}} a^{N_a} b^{N_b} c^{N_c-n}$

$= b^{n(n-1)} \sum_{6V\text{configs}} (a/b)^{N_a} (c/b)^{N_c-n}$

$= b^{n(n-1)} \sum_{ASM} y_{\text{inv}} x_{\text{#}(1)}$

$= b^{n(n-1)} Z_{ASM} (x, y)$

Which we have computed previously

\[ Z_{ASM} (x, y) = \det ((1-u)I + uG)^{(n)} \]
DPP = Arrays of positive integers

Vocabulary
- $a_{ij}$ = part
- $a_{ij} \leq j-i = \text{special part}$
- order $n = a_{ij} \leq n \ \forall i,j$

$\text{OBSERVABLES}$
- $\text{# special parts}$
- $\text{# non-special parts}$

$n=3$

7 DPP's

\[
\begin{array}{cccccccc}
\emptyset & 2 & 3 & 33 & 32 & 31 & 33 & 332 \\
\end{array}
\]
\[ \text{DPP} = \text{Arrays of positive integers} \]

- \[ a_{i,j} = \text{part} \]
- \[ a_{i,j} \leq j-i = \text{special part} \]
- \[ \text{order } n = a_{i,j} \leq n \quad \forall i,j \]

\[ \boxed{n=3: \quad 7 \text{ DPP's}} \]

\[ \emptyset \quad 2 \quad 3 \quad 33 \quad 32 \quad 3 \quad 33 \quad 3 \quad 2 \]

| spec parts | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| non spec parts | 0 | 1 | 1 | 2 | 2 | 2 | 1 | 3 |

\[ \text{compare with ASM} ! \]
ASM-DPP Conjecture

The # of n x n ASM with \((-1) = k\)
and with \(\text{Inv} - #(1) = l\)
is equal to the number of DPP
of order n with \(#\text{ special parts} = k\)
and with \(#\text{ non-special parts} = l\)

(actually even more, but we concentrate on this here).
DPP as Lattice Paths

Cyclically symmetric Rhombus tilings of a Hexagon \((n, n+2, n, n+2, n, n+2)\) with \(\Delta\) hole \(\leftrightarrow\) Lattice Paths

Lalonde '03
(Krattenthaler '06)
DPP as Lattice Paths

Rhombus Tilings of a Hexagon \((n, n+2, n, n+2, n, n+2)\) with \(\Delta\) hole \(\leftrightarrow\) Lattice Paths
DPP as Lattice Paths

(Krattenthaler '06)

Rhombus Tilings of a Hexagon \((n, n+2, n, n+2, n, n+2)\) with \(\Delta\) Hole \(\leftrightarrow\) Lattice Paths
horizontal steps \(-\) = non-special parts

\(n\)

\(n-2\)

\(\begin{align*}
\text{horizontal steps \(-\)} & \quad = \text{special parts} \\
\text{horizontal steps here do not count}
\end{align*}\)
The Gessel–Viennot theorem:

- **Lattice paths** (an oriented graph). Starting vertex \( e_1, e_2, \ldots, e_n \). Ending vertices \( e_1, e_2, \ldots, e_n \).

- Main assumption: if \( i < j \) and \( k < l \), any path \( p_1 \) from \( a_i \to e_e \) and path \( p_2 \) from \( a_j \to e_k \) intersect (i.e., have a common vertex).

- Paths are weighted e.g. \( w(s) \) per step \( s \),

and \( w(p) = \prod_{s \in steps \, p} w(s) \)
Ex: has the crossing property.

**THM [GV]** The partition function for families of $n$ non-intersecting paths starting at $a_1, \ldots, a_n$, ending at $e_1, \ldots, e_n$ is given by:

$$Z_{e_1, \ldots, e_n} = \det (Z_{a_i}^{a_j})_{i,j \leq n}$$

where $Z_a$ = partition function for one path $a \to e$. 
Proof = by involution (cf proof of generalized Bose-Fermion correspondence of chapter 1).

1. Expand \( \text{det} (Z_{a_i}^{e_j}) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \sum_{\text{all paths } a_i \rightarrow e_{\sigma(i)}} \prod \text{step weights} \)

2. Find a config of paths by interchanging the ends of the two paths with "rightmost" crossing.

3. The weights are unchanged except for \( \text{sgn}(\sigma) \rightarrow \text{minus sign} \)

4. We are left with fixed points of \( \phi = \text{Non-Intersecting configs.} \)
horizontal steps — = non-special parts

\begin{align*}
\text{Lemma} & \quad \text{det} \left( I + M \right) = \sum_{n \times n} \text{det}(M_{i_1 \cdots i_k}) \\
\text{Here:} & \quad M_{ij} = \text{Part. funcn (path } (i,0) \rightarrow (0,j))
\end{align*}

\begin{itemize}
\item horizontal steps — = special parts
\item horizontal steps here do not count
\end{itemize}
By Gessel–Viennot theorem

\[ \text{det} (M_{i_1 \cdots i_k}) = \text{Partition function for families of } \]
\[ k \text{ non-intersecting paths starting at } (i_1, 0) \rightarrow (i_k, 0), \text{ ending at } (0, i_1) \cdots (0, i_k) \]

\[ M_{ij} = \text{Partition Function for 1 path} : \]

\[ M_{ij} = \sum \text{paths } (i_0) \rightarrow (0_j) \times \#(\downarrow) \#(\uparrow) \]

horizontal steps: lower wedge, upper wedge
\[ Z_{DPP}^{(n)}(x, y) = \det(I + M) \]

\[ M_{i,j} = \sum_{k=0}^{\frac{i}{j}} \]

pu special part

per non-special part
\[ Z_{DPP}^{(n)}(x, y) = \det(I + M) \]

\[ M_{i,j} = \sum_{k=0}^{l} \sum_{\ell \geq 0} \binom{k}{\ell} \times k - \ell \]
\[ Z_{DPP}^{(n)}(x, y) = \det(I + M) \]

\[ M_{i,j} = \sum_{k=0}^{i} \sum_{l \geq 0} \binom{k}{l} \times (k-l) \binom{j+1}{l} y^{l+1} \]
Generating function:

Trick = we want to have a size n bimatrix rather than size n-1. Make matrix longer by 1 extra row and column

\[ M'_{ij} = M_{i-1,j-1} \quad i,j \geq 0 \]

Note \( M_{-1,j} = 0 \) \( \forall j \geq 0 \), hence:

\[
\det (I + M)^{(n-1)} = \det (I + M')^{(n)}
\]
Generating function:

\[ f_{\text{DPP}}(z,w) = \sum_{i,j \geq 0} z^i w^j (I + M)^i_j \]

**THM**

\[ f_{\text{DPP}}(z,w) = \frac{1}{1 - zw} + \frac{1}{1 - z} \frac{yz}{1 - xzw - (y-x)zw} \]

weights: \( x \) / special part \( y \) / non-special part
**Proof:**

1. Note that we may conjugate \((1-v)I + vG\) by \(\Delta\):

\[
\Delta = \text{diag} \left( x^{i\frac{
abla}{2}} \right)_{i \in \mathbb{Z}_+}
\]

\[
\Delta ((1-v)I + vG) \Delta^{-1} = (1-v)I + vG
\]

where

\[
f_{\tilde{G}}(z,w) = \frac{1}{1-xz-w-(y-x)zw}
\]

and

\[
det((1-v)I + vG)^{(n)} = det((1-v)I + v\tilde{G})^{(n)}
\]
Let \( \tilde{f}_{ASM} = \frac{1 - \nu}{1 - 2 \nu} + \nu \tilde{f}_G(z, w) \).

2. We have the following relation:

\[
(1 - z)(1 - (1 - \nu)w) \tilde{f}_{DPP}(z, w) - (1 - \frac{z}{1 - \nu})(1 - w) \tilde{f}_{ASM}(z, w)
= (x \nu(1 - \nu) - y(1 - \nu) - \nu) \times \text{rational fraction} (z, w)
\]

\[
(x = \left(\frac{q^2 - q^{-2}}{q' r - q r^{-1}}\right), \quad y = \left(\frac{q r - q^{-1} r^{-1}}{q' r - q r^{-1}}\right), \quad \nu = \left(\frac{r^2 - q^2}{q^2 - q^{-2}}\right), \quad 1 - \nu = \left(\frac{q^2 - r^2}{q^2 - q^{-2}}\right)
\]
3. Remark: Let \( A = (A_{ij}) \quad F = \sum_{i,j \geq 0} A_{ij} z^i w^j \)
then \( (1-z)(1-\mu w) F(z,w) \) is the generating function for \( (I-zA)A(I-\mu S)^t \) where
\( S_{ij} = \delta_{ij+}, \) "shift" matrix strictly lower triangular \( \Rightarrow \) the determinant is unchanged in any finite truncation

\( \rightarrow \) The conjecture is proved.
REFINEMENTS

- **ASM**: \# 0's left of 1 in 1st row (ASM)

- **DPP**: \# parts = n

**Z** modifies the last column of the matrix only

\[ Z_{ASM}(x, y, z) = Z_{DPP}(x, y, z) \]  
(MRR)
DOUBLE REFINEMENT

- ASM
- DPP

THM

\[ Z_{ASM}(x, y, z, w) = Z_{DPP}(x, y, z, w) \]

Proof: use Desnanot–Jacobi–Lewis–Carroll to relate \( Z(z, w) \) to \( Z(z, 1) \), \( Z(w, 1) \) and \( Z(1, 1) \)

\[ \begin{array}{c}
\text{HH} \quad \text{HH} \quad = \quad \text{HH} \quad \text{HH} \\
\text{HH} \quad \text{HH} \\
\end{array} \]
For both $Z = Z_{ASM}$ and $Z = Z_{DPP}$:

$$(z-w)Z_N(z,w) Z_{N-1}(1,1) = (z-1)w Z_N(z,1) Z_{N-1}(1,w) - (z-1)w Z_N(1,w) Z_{N-1}(z,1)$$

($z,w$ affects only 2 columns of the matrix).
CONCLUSION

MRR PROVED by method of generating fetsas for the matrix of which we take the det something special about the class of gen fetsas

→ Bijective ASM—DPP? (—TSSCPP?) fermions? (—O(n)?)

→ More refeemenents? other spectral parameters?
Generalizations: DPP with symmetries ← ASM with symmetries [in progress]

q-deformation: $|D| = \Sigma a_{ij}$ for a DPP

$\sum_{D \in DPP(n)} q^{|D|} = \prod_{j=0}^{n-1} \frac{(3j+1)!_q}{(n+j)!_q}$

$p_6 = q$-enumeration of ASMs?