Solving the Yang-Baxter eqn

An equivalent formulation:

\[ \tilde{R}_{12}(z,w) = R_{12}(z,w) P_{12} \]

\[ P_{ij}: V_i \otimes V_j \rightarrow V_j \otimes V_i \]

\[ e_k \otimes e_k \rightarrow e_k \otimes e_k \]

"space permutation operator" (NB does not permute the z, w's)

\[ \tilde{R}_{12} = \]

So YBE becomes:

\[ \tilde{R}_{12} \tilde{R}_{23} \tilde{R}_{12} = \tilde{R}_{23} \tilde{R}_{12} \tilde{R}_{23} \]

\[ (z_1 z_2) (z_1 z_3) (z_2 z_3) (z_2 z_3) (z_1 z_2) (z_1 z_2) \]

Example (6V):

\[ \tilde{R}(z,w) = \]

\[ \begin{bmatrix}
  a(z,w) & 0 & 0 & 0 \\
  0 & c_1(z,w) & b(z,w) & 0 \\
  0 & b(z,w) & c_2(z,w) & 0 \\
  0 & 0 & 0 & a(z,w)
\end{bmatrix} \]
AN IMPORTANT EXERCISE
(introduces naturally Hecke and Temperley–Lieb algebras)

Find the most general solution of the Yang–Baxter + inverse equations of the form

\[ \tilde{R}_{ij}(z,w) = 1 + f(z/w) e_{ij} \]

if smooth fets, \( f(1) = 0 \), \( f'(1) \neq 0 \)

\( e_{ij} \) operator acting on \( V_i \otimes V_j \rightarrow V_i \otimes V_j \)

and with normalization (unitarity condition):

\[ \tilde{R}(z,w) \tilde{R}(w,z) = 1 \]

(separate spectral parameters / spin action)

In particular, find \( f \) and the algebra generated by \( e_i \equiv e_{i,0} \).
1. Start with normalization:

\[(1 + f(z)e)(1 + f(\frac{1}{z})e) = 1\]

\[f(z)f(\frac{1}{z})e^2 + [f(z) + f(\frac{1}{z})]e = 0\]

must be independent of \( z \to 1: f'(1) \neq 0 \Rightarrow \]

\[e^2 = \beta e \quad \text{for some} \quad \beta \in \mathbb{C} \quad \text{fixed,}\]

and \( f \) satisfies \( f(z) + f(\frac{1}{z}) + \beta f(z)f(\frac{1}{z}) = 0 \)

(\( e \) is an un-normalized projector)

2. Write YBE, with \( f = f(z) \quad f' = f(zw) \quad f'' = f(u) \)

and \( e_i \) for \( e_{i+1} \):

\[0 = (1 + f e_1)(1 + f' e_2)(1 + f'' e_1) = (1 + f'' e_2)(1 + f' e_1)(1 + f e_2)\]

\[\text{cst.:} \quad 1 = 1 = 0 \quad \text{OK}\]

\[\text{lin.:} \quad (f'' - f'' + \beta ff')(e_1 - e_2) \quad (\text{use} \quad e^2 = \beta e)\]

\[\text{rad.:} \quad e_1 e_2 (f'f' - f'' f) + e_2 e_1 (f'' f' - f' f'') = 0\]

\[\text{cubic} \quad ffff'' (e_1 e_2 e_1 - e_2 e_1 e_2)\]

again, there exists \( \gamma \) such that:

\[e_1 e_2 e_1 - e_2 e_1 e_2 = \gamma (e_1 - e_2)\]
We may pick \( \gamma = 1 \) (upon redefining \( e \rightarrow e^{\sqrt{\gamma}} \))

\[
ff'f'' - f' + f + f'' + \beta ff'' = 0
\]

\[
f(z) f(zw) f(w) + f(z) f(w) - f(zw) + \beta f(z) f(w) = 0
\]

Taylor expand \( z = 1 + \varepsilon \) \( f'(1) = \alpha \)

\[
3 \alpha f(w)^2 + 3 \alpha + f(w) - f(w) - 3 \varepsilon f'(w) + \beta 3 \alpha \frac{f(w)}{\alpha} = 0
\]

Order 1:

\[
\frac{f^2}{\alpha} + 1 + \beta f = \frac{w f'(w)}{\alpha f(w)}
\]

Write \( \beta = -q - q^{-1} \) then:

\[
(q^2 + 1)(f - q^{-1})(f - q) = \frac{w}{\alpha} f'(w)
\]

\[
\alpha = \frac{w f'}{(q - q^{-1})} \left[ \frac{1}{f - q} - \frac{1}{f - q^{-1}} \right]
\]

\[
(w = e^u) \quad f'(w) = g'(u)
\]

\[
du \alpha(q - q^{-1}) = d \log \left( \frac{g - q}{g - q^{-1}} \right) \quad g'(0) = 0
\]
So finally we integrate this into:

\[
\frac{f - q}{f - q^{-1}} = q e^{\frac{2}{\xi(q-q^{-1})}} \xi \Rightarrow \xi = q \xi
\]

\[
f(1-zq) = q(1-z)
\]

\[
f(z) = \frac{z-1}{qz - q^{-1}}
\]

so we see that this is same as in \(6V\), with

\[
e = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & -q^{-1} & 1 & 0 \\
0 & 1 & -q & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\(e^2 = \beta e, \beta = q+q^{-1}\)

\[
d = 1: \tilde{R}(z,w) = (qz-q^{-1}w)I + (z-w) e_i = \begin{pmatrix}
a & 0 & 0 & 0 \\
0 & c_1 & b & 0 \\
0 & b & c_2 & 0 \\
0 & 0 & 0 & a
\end{pmatrix}
\]

**NB** \(e_1, e_2, \ldots, e_{n-1}\) form a set of generators of the Hecke algebra:

\[
\begin{align*}
& e_i e_{i+1} e_i - e_i = e_{i+1} e_i e_{i+1} - e_{i+1} \\
& e_i^2 = \beta e_i
\end{align*}
\]

Usually better known under simple transformation: \(T_i = q^{1/2} e_i + 1 (= R(z=\infty))\)
then

\[ (1) \quad [q(T_i - 1)]^2 = -(q + q^{-1})(T_i - 1)q^2 \]
\[ T_i^2 = (1 - q^2) T_i + q^2 \]

\[ (2) \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \]

(Braid relation)

NB for the 6V model solution we actually have a quotient of the Hecke algebra by the relations

\[ e_i e_{i+1} e_i = e_i \quad i = 1, 2, \ldots, N-1 \]

This is the Temperley–Lieb algebra.

in the so-called "spin 1/2" representation, namely over \( V = V_1 \otimes V_2 \cdots \otimes V_N \) with \( V_i \cong \mathbb{C}^2 \), with distinguished basis:

\( (e_+, e_-) \).