Section 5.4: Linear Dependence and Independence

Example 1: A paper company produces 3 kinds of paper at 3 different mills. The Huntington mill produces 3 tons of basic white paper, 3 tons of color paper, and 6 tons of stationary paper per day. The Charleston mill produces 6 tons of basic white paper, 8 tons of color paper, and 10 tons of stationary paper per day. The Parkersburg mill produces 9 tons of basic white paper, 10 tons of color paper, and 17 tons of stationary paper per day. A major publishing house has contracted with this company to supply it with paper and the company currently meets that contract by running Huntington for 27 days, Charleston for 23 days and Parkersburg for 30 days. Suddenly a major piece of machinery breaks down in the Parkersburg mill and the mill will need to be shut down for an indefinite period of time. Can the company still meet its contract, and if so, how?
Example 2: Example 1 is an example of redundancy. Because of the linear relationship connecting the three mill vectors, the company has different ways to meet its contract. This is flexibility, a desirable quality of a company’s operations. How would the production schedule look like if the company chose to reduce the production of the Parkersburg mill to $t$ days?

Remark: This is precisely the parametric solution of the system $[ H \ C \ P \mid K ]$. There are infinitely many ways to meet the contract! In linear algebra we refer to this type of redundancy as linear dependence of the mill vectors.

Definition: linear dependence and linear independence
Procedure for Determining whether a Set of Vectors Is Linearly Dependent:

**Example 3:** Are the vectors $\vec{v}_1 = (1, 0, 1, 2)$, $\vec{v}_2 = (0, 1, 1, 2)$, and $\vec{v}_3 = (1, 1, 1, 3)$ in $\mathbb{R}^4$ linearly dependent? If so, find a dependency equation.
Example 4: Are the vectors \( \vec{v}_1 = (1, 2, -1) \), \( \vec{v}_2 = (1, -2, 1) \), \( \vec{v}_3 = (-3, 2, -1) \), and \( \vec{v}_4 = (2, 0, 0) \) in \( \mathbb{R}^3 \) linearly dependent? If so, find a dependency equation.
Theorem: Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ be vectors in $\mathbb{R}^n$. If $k > n$, then these vectors are

Remark:

Let $A$ be an $n \times n$ matrix. If the columns $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ of $A$ are linearly independent, then the equation

$$c_1\vec{a}_1 + c_2\vec{a}_2 + \cdots + c_n\vec{a}_n = \vec{0}$$

has only the one solution:

That is,

$$\begin{bmatrix} A \mid \vec{0} \end{bmatrix} \rightarrow \text{RREF}$$

Theorem: Let $A$ be an $n \times n$ matrix. Then the following two conditions are equivalent:
Example 5: An apparel company makes sweaters, hats, and gloves at four plants. The daily production is as follows:

<table>
<thead>
<tr>
<th></th>
<th>Plant 1</th>
<th>Plant 2</th>
<th>Plant 3</th>
<th>Plant 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>sweaters</td>
<td>100</td>
<td>400</td>
<td>300</td>
<td>700</td>
</tr>
<tr>
<td>hats</td>
<td>200</td>
<td>700</td>
<td>800</td>
<td>800</td>
</tr>
<tr>
<td>gloves</td>
<td>300</td>
<td>1000</td>
<td>1200</td>
<td>1100</td>
</tr>
</tbody>
</table>

The company is contracted to make 4,700 sweaters, 8,600 hats, and 12,500 gloves per week for the Navy. It currently meets this demand by running Plant 1 for 5 days, Plant 2 for 4 days, Plant 3 for 4 days, and Plant 4 for 2 days. Can the company shut down one plant and still meet demand with the others? Give alternate production schedules.
Example 5 (Continued):