Math 125 Fall 2017 PRACTICE Exam 3
Monday, December 4, 2017

Name: ________________________________  NetID: ____________________@illinois.edu
Signature: ________________________________

Circle your lecture:

8am lecture  2pm lecture  3pm lecture

• CODE THE LETTERS BELOW ON YOUR SCANTRON NOW

• Multiple choice questions begin at number 3 on the scantron.

• No cells phones, i-Pods, MP3 players. Turn them off now. If you are seen these items in hand during the exam it will be considered cheating and you will be asked to leave.

• There are 14 question in the exam, 2 of which are free response and 12 are multiple choice.

• In the free response part, you must show all your work to receive full credit.

• Showing your work is NOT required for the multiple choice part.

• You can use your calculator in every problem of this exam.

• No questions will be answered during the exam. If you are uncertain about something, put a note about it on your exam paper and then answer the question as best you can with the information that you are given.

• You have 60 minutes to complete this exam plus 5 minutes extra to fill in your scantron.

• If you finish early, quietly and respectfully get up and hand in your exam.

• When time is up, put down your writing utensil, close your exam, and remain seated. Anyone seen continuing to write after time is called will have their exam marked and lose all points on the page they are writing on.

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DO NOT OPEN EXAM UNTIL TOLD TO DO SO

95. D
96. C
Free-Response score:

1. (20 points) 

2. (20 points) 

Total (40 points): 

Solution. 1. (20 points) The car manufacturers Demon, Speed, and King compete for the nation’s auto trade. Annually, Demon loses 8% of its customers to Speed and 6% to King, while Speed loses 12% of its customers to Demon and 10% to King, and King loses 3% of its customers to Demon 42% to Speed. Initially, Demon has 43%, Speed has 22%, and King has 35% of the market.

A. (6 points) Construct a transition matrix $T$ for the Markov chain above and find the initial state vector $S_0$.

$$T = \begin{bmatrix}
.86 & .12 & .03 \\
.08 & .78 & .42 \\
.06 & .10 & .55 
\end{bmatrix}$$

The state vector $S_0$ is:

$$S_0 = \begin{bmatrix}
.43 \\
.22 \\
.35 
\end{bmatrix}$$

B. (4 points) Find the distribution vector $S_{10}$ after 10 years, then explain the meaning of this vector.

$$S_{10} = T^{10} \cdot S_0 = (0.406804, 0.440786, 0.15241)$$

After 10 years, Demon will have 40.68%, Speed will have 44.08%, and King will have 15.24% of the market.

C. (2 points) Will the market stabilize in less than 10 years (to 4 decimals)? Yes

D. (7 points) Find the exact stable vector $S$ for $T$. Include a detailed step-by-step solution and write down the final answer without rounding. Explain the meaning of this vector.

We want to find a vector $S = (a, b, c)$ that satisfies:

- stability requirement: $(T - I)S = 0$
- stochastic requirement: $a + b + c = 1$

The stable vector for the matrix $T$ is $S = (0.408895, 0.439020, 0.15208)$. Eventually, Demon will have 40.8895%, Speed will have 43.9020%, and King will have 15.208% of the market.

E. (1 point) On the front page of your exam, CLEARLY write down your name and circle your lecture. Fill in your scantron. You will lose 1 point if you fail to bubble in your CORRECT netid, your answers to questions 3–14, or your exam code in your scantron.
Solution. 2. (20 points) An apparel company ‘Winter is coming’ produces hats, gloves, and scarves at four separate factories in accordance with the following table:

<table>
<thead>
<tr>
<th></th>
<th>Factory 1</th>
<th>Factory 2</th>
<th>Factory 3</th>
<th>Factory 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>hats</td>
<td>4</td>
<td>8</td>
<td>44</td>
<td>40</td>
</tr>
<tr>
<td>gloves</td>
<td>3</td>
<td>9</td>
<td>45</td>
<td>36</td>
</tr>
<tr>
<td>scarves</td>
<td>2</td>
<td>3</td>
<td>23</td>
<td>23</td>
</tr>
</tbody>
</table>

They currently fill their total weekly order of 264 hats, 252 gloves, and 139 scarves by running factory 1 for 6 days, factory 2 for 4 days, factory 3 for 2 days, and factory 4 for 3 days.

(A) (2 points) By filling in the blanks below, write out the linear combination that represents the current production schedule to fill this contract.

\[ \text{Contract} = \begin{pmatrix} 6 \\ 4 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} \text{F}_1 \\ \text{F}_2 \\ \text{F}_3 \\ \text{F}_4 \end{pmatrix} \]

(B) (6 points) Solve for, and write out a dependency equation for the factories.

Our task is to find coefficients \( c_1, c_2, c_3, c_4 \), not all zero, such that \( c_1 \text{F}_1 + c_2 \text{F}_2 + c_3 \text{F}_3 + c_4 \text{F}_4 = \vec{0} \).

\[
\begin{bmatrix}
\text{F}_1 & \text{F}_2 & \text{F}_3 & \text{F}_4 & \text{0} \\
4 & 3 & 2 & 1 & 0 \\
8 & 9 & 3 & 0 & 0 \\
44 & 45 & 23 & -2 & 0 \\
40 & 36 & 23 & 1 & 1 \\
\end{bmatrix}
\begin{pmatrix}
c_1 \\
c_2 \\
c_3 \\
c_4 \\
\end{pmatrix}
= 0
\]

We introduce a parameter \( t \) for the free variable \( c_4 \). Then \( c_1 = -3t, \ c_2 = 2t, \ c_3 = -t, \) and \( c_4 = t \). Setting \( t = 1 \) leads to the following dependency equation:

\[ -3\text{F}_1 + 2\text{F}_2 - \text{F}_3 + \text{F}_4 = \vec{0} \]

(C) (4 points) Write out a linear combination for each factory.

\[
\begin{align*}
\text{F}_1 &= (2/3)\text{F}_2 - (1/3)\text{F}_3 + (1/3)\text{F}_4 \\
\text{F}_2 &= (3/2)\text{F}_1 + (1/2)\text{F}_3 - (1/2)\text{F}_4 \\
\text{F}_3 &= -3\text{F}_1 + 2\text{F}_2 + \text{F}_4 \\
\text{F}_4 &= 3\text{F}_1 - 2\text{F}_2 + \text{F}_3 \\
\end{align*}
\]

(D) (8 points) Can the company shut down one of their factories and meet their current contract with the other three factories? If so, give alternate production schedules. Otherwise, explain why each of the factories cannot be shut down. (Use the equations above, and the equation in part (A) to answer this question).

**Closing Factory 1**

\[
\text{Contract} = 6\text{F}_1 + 4\text{F}_2 + 2\text{F}_3 + 3\text{F}_4
\]

\[= 6(2/3\text{F}_2 - 1/3\text{F}_3 + 1/3\text{F}_4) + 4\text{F}_2 + 2\text{F}_3 + 3\text{F}_4
\]

\[= 8\text{F}_2 + 5\text{F}_4
\]

No, we can’t close Factory 1 because we would have to run Factory 2 for 8 days a week.

**Closing Factory 2**
\[
\overrightarrow{C} = \overrightarrow{6F_1} + 4\overrightarrow{F_2} + 2\overrightarrow{F_3} + 3\overrightarrow{F_4}
\]
\[
\overrightarrow{C} = \overrightarrow{6F_1} + 4(3/2F_1 + 1/2F_3 - 1/2F_4) + 2F_3 + 3F_4
\]
\[
\overrightarrow{C} = 12\overrightarrow{F_1} + 4\overrightarrow{F_3} + 3\overrightarrow{F_4}
\]

No, we can’t close Factory 2 because we would have to run Factory 2 for 12 days a week.

Closing Factory 3

\[
\overrightarrow{C} = \overrightarrow{6F_1} + 4\overrightarrow{F_2} + 2\overrightarrow{F_3} + 3\overrightarrow{F_4}
\]
\[
\overrightarrow{C} = \overrightarrow{6F_1} + 4\overrightarrow{F_2} + 2(-3\overrightarrow{F_1} + 2\overrightarrow{F_2} + \overrightarrow{F_4}) + 3\overrightarrow{F_4}
\]
\[
\overrightarrow{C} = 8\overrightarrow{F_2} + 5\overrightarrow{F_4}
\]

No, we can’t close Factory 3 because we would have to run Factory 2 for 8 days a week.

Closing Factory 4

\[
\overrightarrow{C} = \overrightarrow{6F_1} + 4\overrightarrow{F_2} + 2\overrightarrow{F_3} + 3\overrightarrow{F_4}
\]
\[
\overrightarrow{C} = \overrightarrow{6F_1} + 4\overrightarrow{F_2} + 2\overrightarrow{F_3} + 3(3\overrightarrow{F_1} - 2\overrightarrow{F_2} + \overrightarrow{F_3})
\]
\[
\overrightarrow{C} = 15\overrightarrow{F_1} - 2\overrightarrow{F_2} + 5\overrightarrow{F_4}
\]

No, we can’t close Factory 4 because we would have to run Factory 1 for 15 days a week and Factory 2 for negative number of days.

How would the answers change if our contract was a monthly contract (assume 30 days in a month)?

- We would be able to close Factory 1 (run Factory 2 for 8 days a month and Factory 4 for 5 days a month).
- We would be able to close Factory 2 (run Factory 1 for 12 days, Factory 3 for 4 days, and Factory 4 for 1 day a month).
- We would be able to close Factory 3 (run Factory 2 for 8 days, and Factory 4 for 5 days a month).
- We would NOT be able to close Factory 4 because we can’t run Factory 2 for negative number of days.
3. (5 points) A home cleaning consumer market in Urbana, Illinois, involves a competition for market shares among three home cleaning companies: Merry Maids (MM), Cleaning Masters (CM) and Services With a Smile (SWS). A consumer survey revealed that at the moment Merry Maids has 10% of the market, Cleaning Masters has 15%, and Service With a Smile has 75%. Each month 10% of Merry Maids customers change to Cleaning Masters and 60% of Merry Maids customers change to Services With a Smile, 25% of Cleaning Masters customers change to Merry Maids and 70% of Cleaning Masters customers change to Services With a Smile, and 20% of Service With a Smile customers change to Merry Maids and 15% of Service With a Smile customers change to Cleaning Masters.

Which of the following is a transition matrix for this problem?

\[
\begin{bmatrix}
0.10 & 0.20 & 0.25 \\
0.30 & 0.15 & 0.05 \\
0.60 & 0.65 & 0.70
\end{bmatrix}
\]

(A) \hspace{1cm} MM CM SWS

\[
\begin{bmatrix}
0.60 & 0.20 & 0.25 \\
0.30 & 0.15 & 0.05 \\
0.10 & 0.65 & 0.70
\end{bmatrix}
\]

(B) \hspace{1cm} MM CM SWS

\[
\begin{bmatrix}
0.30 & 0.25 & 0.20 \\
0.10 & 0.05 & 0.15 \\
0.60 & 0.70 & 0.65
\end{bmatrix}
\]

(C) \hspace{1cm} \star

\[
\begin{bmatrix}
0.30 & 0.20 & 0.20 \\
0.20 & 0.10 & 0.05 \\
0.60 & 0.75 & 0.75
\end{bmatrix}
\]

(D) \hspace{1cm} MM CM SWS

\[
\begin{bmatrix}
0.10 & 0.20 & 0.70 \\
0.30 & 0.15 & 0.05 \\
0.60 & 0.65 & 0.25
\end{bmatrix}
\]

(E) \hspace{1cm} MM CM SWS

Solution. \( T = \)

\[
\begin{bmatrix}
0.30 & 0.25 & 0.20 \\
0.10 & 0.05 & 0.15 \\
0.60 & 0.70 & 0.65
\end{bmatrix}
\]

\( \text{MM} \hspace{1cm} CM \hspace{1cm} SWS \)
4. (5 points) Let

\[ A = \begin{bmatrix} 5 & 1 \\ -8 & 3 \\ 1 & -2 \end{bmatrix} \]

Out of the following vectors:
\[ \vec{u} = (7, -2, -3) \]
\[ \vec{v} = (4, -13, -5) \]
\[ \vec{w} = (-15, 1, 8) \]

which are in the column space of \( A \)?

(A) \( \vec{w} \) only
(B) \( \vec{u} \) and \( \vec{v} \)
(C) All of these vectors are in the column space of \( A \)
(D) \( \star \) \( \vec{u} \) and \( \vec{w} \)
(E) None of these vectors are in the column space of \( A \)

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Solution. -
5. (5 points) Let

\[ S = \{(1, -2, 8), (5, 1, -15), (-8, 3, 1)\} \]

Give a geometric description of the space that \( S \) spans.

(A) All of \( \mathbb{R}^2 \)
(B) \( \star \) A plane in \( \mathbb{R}^3 \) passing through the origin
(C) \( \star \) A plane in \( \mathbb{R}^3 \) passing through \((1, -2, 8)\)
(D) A line in \( \mathbb{R}^3 \) passing through the origin
(E) All of \( \mathbb{R}^3 \)

**Solution.** In your actual exam, always only one of the choices should be correct. I made a mistake here which is why there are two correct answers.

If it happens that in an actual exam there is a problem where more than one answer appears to be right, choose the answer that seems most correct. You can also add a note in the exam paper explaining why you think more than one choices are correct (or why you think none of the choices are correct for example).
6. (5 points) What is the null space of the following matrix $A$?

$$
A = \begin{bmatrix}
3 & -2 & 4 \\
-1 & 1 & -3 \\
4 & -2 & 2
\end{bmatrix}
$$

(A) span\{(3, -2, 4)\}

(B) \{(0, 0, 0)\}

(C) All of $\mathbb{R}^3$.

(D) $\ast$ span\{(2, 5, 1)\}

(E) span\{(3, -2, 4), (-1, 1, -3)\}

Solution. -
7. (5 points) Which of the following is NOT a line in \( \mathbb{R}^3 \)?

(A) ★ the set of points \((x, y, z)\) such that \(3x - 5y = 4\)
(B) \(\text{span}\{(1, -2, 1), (-3, 6, -3)\}\)
(C) the set of solutions \((x, y, z)\) of the system of equations \(x + 2y - 5z = 3\) and \(-x + y + 3z = 1\)
(D) \(\{(3t - 1, t, 2t + 4) \mid t \in \mathbb{R}\}\)
(E) the set of points \((x, y, z) = (1, 1, 2) + \text{span}\{(1, -2, 1)\}\)

Solution. -
8. (5 points) A paper company produces 3 kinds of paper at 3 different mills. The Huntington mill produces 3 tons of basic white paper, 3 tons of color paper, and 6 tons of stationary paper per 8-hour work day. The Charleston mill produces 6 tons of basic white paper, 8 tons of color paper, and 10 tons of stationary paper per 8-hour work day. The Parkersburg mill produces 9 tons of basic white paper, 10 tons of color paper, and 17 tons of stationary paper per 8-hour work day. The Huntington mill is expensive to run, so the company is exploring shutting it down. Is it possible that the other two mills can be run to exactly make up for the daily output lost by shutting down the Huntington mill?

(A) No, because we would need to run one of the other mills for negative numbers of hours.

(B) No, because the production vector of Huntington mill cannot be expressed as a linear combination of the other two production vectors.

(C) Yes. To replace the Huntington mill, the Parkersburg mill must be run for 16 hours and the Charleston mill for 20 hours a day.

(D) ★ Yes. To replace the Huntington mill, the Parkersburg mill must be run for 12 hours and the Charleston mill for 6 hours a day.

(E) No, because we would need to run one of the other mills for more than 24 hours.

Solution. -
9. (5 points) Let \( S \) be the set of vectors \( u = (-1, -3, 1) \), \( v = (1, 0, 1) \), and \( w = (1, 2, 3) \). Which of the following claims is FALSE?

(A) ★ The span of \( S \) is a plane passing through the origin

(B) The matrix whose columns are the vectors \( u \), \( v \), and \( w \) is invertible.

(C) The span of \( S \) is a subspace of \( \mathbb{R}^3 \)

(D) The set \( S = \{u, v, w\} \) is linearly independent

(E) The point \((6, 12, -5)\) can be expressed as a linear combination of \( u \), \( v \), and \( w \).

Solution. -
10. (5 points) Sun city annually loses 35% of its residents to the suburbs, while the suburbs lose 45% of their residents to the city. Sun City currently has 65% of the population. What is the exact stable vector of this system?

(A) \[
\begin{bmatrix}
.55 \\
.45
\end{bmatrix}
\]

(B) \[
\begin{bmatrix}
.65 \\
.35
\end{bmatrix}
\]

(C) \[
\begin{bmatrix}
.5625 \\
.4375
\end{bmatrix}
\]

(D) The correct answer is not here.

(E) \[
\begin{bmatrix}
.5825 \\
.4175
\end{bmatrix}
\]

Solution. -
11. (5 points) Consider below the transition matrix $T$. How many years will it take to stabilize at the stable distribution if we work only to two decimal places?

\[
T = \begin{bmatrix}
.75 & .25 \\
.25 & .75
\end{bmatrix}
\]

(A) ⭐ 7 years.
(B) 9 years.
(C) 4 years.
(D) The correct answer is not here.
(E) 6 years.

Solution.

-
12. (5 points) The transition matrix of a certain Markov chain/process is

\[ T = \begin{bmatrix} .2 & .3 \\ .8 & .7 \end{bmatrix} \]

Find the exact stable vector of the process among the list:

(A) The correct answer is not here.

(B) \( \begin{bmatrix} 4/11 \\ 7/11 \end{bmatrix} \)

(C) \( \begin{bmatrix} 6/11 \\ 5/11 \end{bmatrix} \)

(D) \( \begin{bmatrix} 5/11 \\ 6/11 \end{bmatrix} \)

(E) \( \star \begin{bmatrix} 3/11 \\ 8/11 \end{bmatrix} \)

Solution.

\[
\begin{bmatrix} -.8 & .3 & 0 \\ 1 & 1 & 1 \end{bmatrix} \Rightarrow RREF = \begin{bmatrix} 1 & 0 & 3/11 \\ 0 & 1 & 8/11 \end{bmatrix}
\]
13. (5 points) A trash service consumer market involves a competition for market shares among two auto service firms: Republic Services and Advanced Disposal. A consumer survey has revealed that each month 20% of Republic Services’ customers change to Advanced Disposal and 44% of Advanced Disposal’s customers change to Republic Services. The approximate stable vector (to four decimal accuracy) for this process is

\[ \vec{S}_{\text{stable}} = \begin{bmatrix} 0.6875 \\ 0.3125 \end{bmatrix} \]

If \( k \) is the month at which this process stabilizes, then

(A) ★ 9 ≤ \( k \) ≤ 11
(B) 6 ≤ \( k \) ≤ 8
(C) The correct answer is not here.
(D) 12 ≤ \( k \) ≤ 14
(E) 15 ≤ \( k \) ≤ 17

Solution. The transition matrix and stable matrix of this process are

\[ T = \begin{bmatrix} 0.8 & 0.44 \\ 0.2 & 0.56 \end{bmatrix}, SM = \begin{bmatrix} 0.7586 & 0.7586 \\ 0.3125 & 0.3125 \end{bmatrix} \]

Also

\[ T^9 = \begin{bmatrix} 0.6875 & 0.6874 \\ 0.3125 & 0.3126 \end{bmatrix} \text{ and } T^{10} = \begin{bmatrix} 0.6875 & 0.6875 \\ 0.3125 & 0.3125 \end{bmatrix} \]

so \( k = 10 \).
14. (5 points) Consider the following information about nutritional ingredients and cost that are important in cottage cheese, cultured milk, swiss cheese, and yogurt per one ounce of each:

<table>
<thead>
<tr>
<th></th>
<th>cottage cheese</th>
<th>cultured milk</th>
<th>swiss cheese</th>
<th>yogurt</th>
</tr>
</thead>
<tbody>
<tr>
<td>calories</td>
<td>25</td>
<td>60</td>
<td>110</td>
<td>15</td>
</tr>
<tr>
<td>protein (g)</td>
<td>4</td>
<td>1</td>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td>lysine (mg)</td>
<td>300</td>
<td>60</td>
<td>510</td>
<td>90</td>
</tr>
<tr>
<td>cost (cents)</td>
<td>7</td>
<td>4</td>
<td>19</td>
<td>7</td>
</tr>
</tbody>
</table>

Suppose that we want to use cottage cheese, cultured milk, and yogurt to make an ‘artificial swiss cheese’ that has the same amount of calories, protein, and lysine as swiss cheese. Artificial swiss cheese can be made if and only if there exist

(A) non-negative numbers $c_1$, $c_2$, $c_3$ such that $c_1(25, 60, 110, 15) + c_2(4, 1, 7, 1) + c_3(300, 60, 510, 90) = (7, 4, 19, 7)$

(B) real numbers $c_1$, $c_2$, $c_3$ such that $c_1(25, 4, 300, 7) + c_2(60, 1, 60, 4) + c_3(15, 1, 90, 7) = (110, 7, 510, 19)$

(C) non-negative numbers $c_1$, $c_2$, $c_3$ such that $c_1(25, 4, 300, 7) + c_2(60, 1, 60, 4) + c_3(15, 1, 90, 7) = (110, 7, 510, 19)$

(D) real numbers $c_1$, $c_2$, $c_3$ such that $c_1(25, 60, 110, 15) + c_2(4, 1, 7, 1) + c_3(300, 60, 510, 90) = (7, 4, 19, 7)$

(E) non-negative numbers $c_1$, $c_2$, $c_3$ such that $c_1(25, 4, 300) + c_2(60, 1, 60) + c_3(15, 1, 90) = (110, 7, 510)$

Solution. -