Name: ________________________________  NetID: ____________________________@illinois.edu

Signature: ________________________________

Circle your lecture:

8am lecture  2pm lecture  3pm lecture

• CODE THE LETTERS BELOW ON YOUR SCANTRON NOW
  • Multiple choice questions begin at number 3 on the scantron.
  • No cells phones, i-Pods, MP3 players. Turn them off now. If you are seen these items in hand during the exam it will be considered cheating and you will be asked to leave.
  • There are 14 question in the exam, 2 of which are free response and 12 are multiple choice.
  • In the **free response** part, you must **show all your work** to receive full credit.
  • Showing your work is NOT required for the multiple choice part.
  • You can use your calculator in every problem of this exam.
  • **No questions** will be answered during the exam. If you are uncertain about something, put a note about it on your exam paper and then answer the question as best you can with the information that you are given.
  • You have **60 minutes** to complete this exam plus **5 minutes** extra to fill in your scantron.
  • If you finish early, quietly and respectfully get up and hand in your exam.
  • When time is up, put down your writing utensil, close your exam, and remain seated. Anyone seen continuing to write after time is called will have their exam marked and lose all points on the page they are writing on.

**DO NOT OPEN EXAM UNTIL TOLD TO DO SO**

89. B
90. A
91. A
92. A
93. A
94. E
95. D
96. A
Free-Response score:

1. (20 points) ________________

2. (20 points) ________________

Total (40 points): ____________
1. (20 points) A transportation survey reveals that 10% of residents of Champaign-Urbana use the bus to commute, 80% drive a car, and 10% ride a bike. In order to reduce gas emissions, the city decides to dramatically increase parking fees and improve bike paths. As a result of these changes, every year, 5% of bus passengers switch to driving a car and 5% of bus passengers switch to riding a bike, 5% of those driving cars switch to the bus and 10% of car drivers switch to riding a bike, and 3% of bikers switch to using a bus and 2% of bikers switch to using a car.

A. (5 points) Construct a transition matrix $T$ and the initial distribution $S_0$ for the Markov chain above.

$T = \begin{bmatrix}
\text{Bus} & \text{Car} & \text{Bike} \\
\text{switch to bus} & \text{switch to car} & \text{switch to bike}
\end{bmatrix}$

$S_0 = \begin{bmatrix}
\text{bus} \\
\text{car} \\
\text{bike}
\end{bmatrix}$

B. (4 points) Find the distribution $S_{11}$ after 11 years and interpret what it tells you.

C. (4 points) The population of Champaign-Urbana is 130,000.

- How many residents of C-U commute by car initially?
- How many residents of C-U commute by car after 11 years?

D. (6 points) Find the exact stable vector $S$ for $T$. Include a detailed step-by-step solution and write down the final answer without rounding.

E. (1 point) On the front page of your exam, CLEARLY write down your name and circle your lecture. In your scantron, bubble in your CORRECT netid, your answers to questions 3–14, and exam code.
Solution. A transportation survey reveals that 10% of residents of Champaign-Urbana use the bus to commute, 80% drive a car, and 10% ride a bike. In order to reduce gas emissions, the city decides to dramatically increase parking fees and improve bike paths. As a result of these changes, every year, 5% of bus passengers switch to driving a car and 5% of bus passengers switch to riding a bike, 5% of those driving cars switch to the bus and 10% of car drivers switch to riding a bike, and 3% of bikers switch to using a bus and 2% of bikers switch to using a car.

A. (5 points) Construct a transition matrix $T$ and the initial distribution $S_0$ for the Markov chain above.

$$T = \begin{bmatrix} .9 & .05 & .03 \\ .05 & .85 & .02 \\ .05 & .1 & .95 \end{bmatrix}$$

- switch to bus
- switch to car
- switch to bike

$$S_0 = \begin{bmatrix} .1 \\ .8 \\ .1 \end{bmatrix}$$

B. (4 points) Find the distribution $S_{11}$ after 11 years and interpret what it tells you.

$$S_{11} = T^{11} \cdot S_0 = (0.2490, 0.2391, 0.5119)$$

After 11 years, 24.90% residents of C-U will use the bus to commute, 23.91% will drive a car, and 51.19% will ride a bike.

C. (4 points) The population of Champaign-Urbana is 130,000.

- How many residents of C-U commute by car initially? 104,000 (80% of 130,000)
- How many residents of C-U commute by car after 11 years? 31,083 (23.91% of 130,000)

D. (6 points) Find the exact stable vector for $T$. Include a step-by-step solution and write down the final answer without rounding.

We want to find a vector $S = (a, b, c)$ that satisfies:

- stability requirement: $(T - I)S = 0$
- stochastic requirement: $a + b + c = 1$

$$[T - I \mid 0] = \begin{bmatrix} -1 & .05 & .03 & 0 \\ .05 & -1.15 & .02 & 0 \\ .05 & .1 & -1.05 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -.1 & .05 & .03 & 0 \\ .05 & -1.15 & .02 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The last equation is redundant so we can disregard it. We need to find $a, b, c$ such that satisfy the following system of equations

$$-1.1a + .05b + .03c = 0$$
$$0.05a - 1.15b + .02c = 0$$
$$a + b + c = 1$$

The stable vector $S$ for the matrix $T$ is $S = (11/43, 7/43, 25/43)$. 
E. (1 point) On the front page of your exam, CLEARLY write down your name and circle your lecture. In your scantron, bubble in your CORRECT netid, your answers to questions 3–14, and exam code.
2. (20 points) A company produces party hats, noisemakers, and garlands at four factories. The daily production is shown in the following table:

<table>
<thead>
<tr>
<th></th>
<th>Factory 1</th>
<th>Factory 2</th>
<th>Factory 3</th>
<th>Factory 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Noisemakers</td>
<td>3</td>
<td>45</td>
<td>9</td>
<td>36</td>
</tr>
<tr>
<td>Party Hats</td>
<td>2</td>
<td>23</td>
<td>3</td>
<td>23</td>
</tr>
<tr>
<td>Garlands</td>
<td>2</td>
<td>22</td>
<td>4</td>
<td>20</td>
</tr>
</tbody>
</table>

They currently fill their total weekly order by running Factory 1 for 3 days, Factory 2 for 2 days, Factory 3 for 2 days, and Factory 4 for 1 day.

A. (2 points) Find the linear combination that represents the current production schedule to fill this contract.

\[ C = F_1 + F_2 + F_3 + F_4 \]

B. (6 points) Find a dependency equation for the factories.

\[ F_1 + F_2 + F_3 + F_4 = 0 \]

C. (4 points) Write out a linear combination for Factory 1 and Factory 2.

\[ F_1 = F_2 + F_3 + F_4 \]

\[ F_2 = F_1 + F_3 + F_4 \]

D. (4 points) Can the company shut down Factory 1 and meet their current contract with the other three factories? If so, give the new production schedule. Otherwise, explain why not.

E. (4 points) Can the company shut down Factory 2 and meet their current contract with the other three factories? If so, give the new production schedule. Otherwise, explain why not.
Solution. A company produces party hats, noisemakers, and garlands at four factories. The daily production is shown in the following table:

<table>
<thead>
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<th></th>
<th>Factory 1</th>
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<td>23</td>
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<td>23</td>
</tr>
<tr>
<td>Garlands</td>
<td>2</td>
<td>22</td>
<td>4</td>
<td>20</td>
</tr>
</tbody>
</table>

They currently fill their total weekly order by running Factory 1 for 3 days, Factory 2 for 2 days, Factory 3 for 2 days, and Factory 4 for 1 day.

A. (2 points) By filling in the blanks below, write out the linear combination that represents the current production schedule to fill this contract.

\[
\text{Contract} = C = 3F_1 + 2F_2 + 2F_3 + F_4
\]

B. (6 points) Find a dependency equation for the factories.

Our task is to find coefficients \( c_1, c_2, c_3, c_4 \), not all zero, such that

\[
\begin{bmatrix}
F_1 & F_2 & F_3 & F_4 \hline
3 & 45 & 9 & 36 \\
2 & 23 & 3 & 23 \\
2 & 22 & 4 & 20
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 3 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & -2 & 0
\end{bmatrix}
\]

We introduce a parameter \( t \) for the free variable \( c_4 \). Then \( c_1 = -3t \), \( c_2 = -t \), \( c_3 = 2t \), and \( c_4 = t \). Setting \( t = 1 \) leads to the following dependency equation:

\[-3F_1 - F_2 + 2F_3 + F_4 = 0.\]

C. (4 points) Write out a linear combination for Factory 1 and Factory 2.

\[
F_1 = (\frac{-1}{3})F_2 + (\frac{2}{3})F_3 + \frac{1}{3}F_4
\]

\[
F_2 = -3F_1 + 2F_3 + F_4
\]

D. (4 points) Can the company shut down Factory 1 and meet their current contract with the other three factories? If so, give the new production schedule. Otherwise, explain why not.

\[
C = 3F_1 + 2F_2 + 2F_3 + F_4
\]

\[
= 3(-\frac{1}{3}F_2 + \frac{2}{3}F_3 + \frac{1}{3}F_4) + 2F_2 + 2F_3 + F_4
\]

\[
= F_2 + 4F_3 + 2F_4
\]

Yes, we can close Factory 1. New production schedule: To meet the contract, we need to run Factory 2 for 1 day a week, Factory 3 for 4 days a week, and Factory 4 for 2 days a week.

E. (4 points) Can the company shut down Factory 2 and meet their current contract with the other three factories? If so, give the new production schedule. Otherwise, explain why not.

\[
C = 3F_1 + 2F_2 + 2F_3 + F_4
\]

\[
= 3F_1 + 2(-3F_1 + 2F_3 + F_4) + 2F_3 + F_4
\]

\[
= -3F_1 + 6F_3 + 3F_4
\]

No, we can’t close Factory 2 because we would have to run Factory 1 for negative days.
3. (5 points) Let \[ S = \{(4, 2, 6), (5, -1, 3), (2, 8, 12)\}. \]

Out of the following vectors:
\[ \vec{u} = (6, 3, 9) \]
\[ \vec{v} = (12, -1, 9) \]
\[ \vec{w} = (2, 7, 3) \]

which are in the span of \( S \)?

(A) only \( \vec{u} \) and \( \vec{w} \)
(B) \( \vec{u}, \vec{v}, \) and \( \vec{w} \)
(C) only \( \vec{w} \)
(D) ★ only \( \vec{u} \) and \( \vec{v} \)
(E) only \( \vec{u} \)

Solution.

\[
\begin{bmatrix}
4 & 5 & 2 & 6 & 12 & 2 \\
2 & -1 & 8 & 3 & -1 & 7 \\
6 & 3 & 12 & 9 & 9 & 3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 3 & 3/2 & 1/2 & 0 \\
0 & 1 & -2 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

The system with \( \vec{u} \) on the RHS is consistent, so \( \vec{u} \) is in the span of \( S \).
The system with \( \vec{v} \) on the RHS is consistent, so \( \vec{v} \) is in the span of \( S \).
The system with \( \vec{w} \) on the RHS is NOT consistent, so \( \vec{w} \) is NOT in the span of \( S \).
4. (5 points) Consider the following information about nutritional ingredients and cost that are important in cottage cheese, cultured milk, traditional swiss cheese, and yogurt per one ounce of each:

<table>
<thead>
<tr>
<th></th>
<th>cottage cheese</th>
<th>cultured milk</th>
<th>traditional swiss cheese</th>
<th>yogurt</th>
</tr>
</thead>
<tbody>
<tr>
<td>calories</td>
<td>25</td>
<td>60</td>
<td>110</td>
<td>15</td>
</tr>
<tr>
<td>protein (g)</td>
<td>4</td>
<td>1</td>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td>lysine (mg)</td>
<td>300</td>
<td>60</td>
<td>510</td>
<td>90</td>
</tr>
<tr>
<td>cost (cents)</td>
<td>5</td>
<td>10</td>
<td>30</td>
<td>15</td>
</tr>
</tbody>
</table>

We want to use cottage cheese, cultured milk, and yogurt to make synthetic swiss cheese that has the same amount of calories, protein, and lysine as the traditional swiss cheese. How much would it cost to produce a piece of synthetic swiss cheese that has 220 calories?

(A) ★ 44 cents

(B) It is impossible to make a product nutritionally equivalent to the traditional swiss cheese from cottage cheese, cultured milk, and yogurt.

(C) 46 cents

(D) 40 cents

(E) 42 cents

**Solution.** If we can find non-negative constants $c_1, c_2, c_3$ such that

\[c_1(25, 4, 300) + c_2(60, 1, 60) + c_3(15, 1, 90) = (110, 7, 510),\]

then we can make synthetic swiss cheese using $c_1$ oz of cottage cheese, $c_2$ oz of cultured milk, and $c_3$ oz of yogurt. The solution to this system is $c_1 = 7/5, \ c_2 = 6/5, \ c_3 = 1/5$.

Cost to produce 110 calories worth of synthetic swiss cheese: \((7/5) \cdot 5 + (6/5) \cdot 10 + (1/5) \cdot 10 = 22\)

Cost to produce 220 calories worth of synthetic swiss cheese: \(2 \cdot 22 = 44\)
5. (5 points) A home cleaning consumer market in Urbana, Illinois, involves a competition for market shares among two home cleaning companies: Merry Maids (MM) and Cleaning Masters (CM). A consumer survey revealed that at the moment Merry Maids has 18% of the market and Cleaning Masters has 82%. Moreover, each month 22% of Merry Maids customers change to Cleaning Masters and 33% of Cleaning Masters customers change to Merry Maids.

What are the market shares of the two cleaning services at the end of month five?

(A) ★ 59.22% for Merry Maids and 40.78% for Cleaning Masters
(B) The correct answer is not here.
(C) 56.17% for Merry Maids and 43.83% for Cleaning Masters
(D) 51.50% for Merry Maids and 48.50% for Cleaning Masters
(E) 58.28% for Merry Maids and 41.72% for Cleaning Masters

Solution. The transition matrix, initial market share vector, and five month market share vector are

\[
T = \begin{bmatrix}
.78 & .33 \\
.22 & .67
\end{bmatrix}, \quad \vec{S}_0 = \begin{bmatrix}
.18 \\
.82
\end{bmatrix}, \quad \vec{S}_5 = \begin{bmatrix}
.78 & .33 \\
.22 & .67
\end{bmatrix}^5 \begin{bmatrix}
.18 \\
.82
\end{bmatrix} = \begin{bmatrix}
.5922 \\
.4078
\end{bmatrix}
\]
6. (5 points) Out of the following:

(I) \{ (x_1, x_2, x_3) : x_1, x_2, and x_3 are all real numbers, x_3 \geq 0 \}

(II) \{ (0, x_2, x_3) : x_2 and x_3 are all real numbers \}

(III) \{ (t, 2t, 1 - t) : t is a real number \}

which are subspaces of \( \mathbb{R}^3 \)?

(A) ★ Only (II)

(B) Only (I) and (III)

(C) Only (II) and (III)

(D) Only (III)

(E) (I), (II), and (III) are all subspaces of \( \mathbb{R}^3 \)

Solution.

- \( S_1 = \{ (x_1, x_2, x_3) : x_1, x_2, and x_3 are all real numbers, x_3 \geq 0 \} \) is not closed under scalar multiplication. If you multiply a point \( (x_1, x_2, x_3) \) from the set \( S_1 \) by a negative number \( c \), the last coordinate \( cx_3 \) will be negative so the resulting point \( (cx_1, cx_2, cx_3) \) is not in \( S_1 \) (every point in \( S_1 \) must have a non-negative last coordinate). Thus, \( S_1 \) is NOT a subspace.

- \( S_2 = \{ (0, x_2, x_3) : x_2 and x_3 are all real numbers \} \) is a subspace of \( \mathbb{R}^3 \). You can check that \( S_2 \) is closed under addition and scalar multiplication. Or you can observe that \( S_2 \) is a span of \( (0, 1, 0) \) and \( (0, 0, 1) \). This is because \( (0, x_2, x_3) = x_2(0, 1, 0) + x_3(0, 0, 1) \) where \( x_2 \) and \( x_3 \) are any real numbers. Every span is a subspace, so \( S_2 \) is a subspace.

- \( S_3 = \{ (t, 2t, 1 - t) : t is a real number \} \) does not contain the origin, so it is NOT a subspace.
7. (5 points) Four of the following five statements are all equivalent to each other. Choose the one statement that is not equivalent to the other four.

(A) the row rank of \( A \) is equal to the row rank of \([A|b]\)

(B) The system \( Ax = b \) is consistent.

(C) ★ The reduced row echelon form of \([A|b]\) has no zero row.

(D) \( b \) is in the column space of \( A \).

(E) \( b \) can be expressed as a linear combination of the columns of \( A \).

Solution. -
8. (5 points) The null space of a matrix $A$ is the set of all solutions $\vec{x}$ such that $A\vec{x} = \vec{0}$. Let

$$A = \begin{bmatrix} -1 & -1 & 4 \\ 3 & 1 & -2 \\ -4 & -2 & 6 \end{bmatrix}$$

What is the null space of $A$?

(A) All of $\mathbb{R}^3$.

(B) $\{(0, 0, 0)\}$

(C) $\text{span}\{(-1, -1, 4), (3, 1, -2)\}$

(D) $\star \text{span}\{(-1, 5, 1)\}$

(E) $\text{span}\{(4, 0, 1)\}$

---

**Solution.**

$$A = \begin{bmatrix} -1 & -1 & 4 & 0 \\ 3 & 1 & -2 & 0 \\ -4 & -2 & 6 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The third column does not contain a leading 1, so $z$ is a free variable and we introduce a parameter $z = t$, where $t$ is any real number. The first two rows give $x + t = 0$ and $y - 5t = 0$. Thus $x = -t$, $y = 5t$, $z = t$. It follows that the set of solutions $\vec{x} = (x, y, z)$ to the system $A\vec{x} = \vec{0}$ is

$$\{(-t, 5t, t) \mid t \in \mathbb{R}\} = \text{span}\{(-1, 5, 1)\}$$
9. (5 points) An auto service consumer market in Champaign, Illinois, involves a competition for market shares among two auto service firms: Car-X Tire and Auto (CarX) and Ricks Automotive Services (RAS). A consumer survey has revealed that each month 14% of Car-X customers change to Ricks Automotive Services and 44% of Ricks Automotive customers change to Car-X Tire and Auto. The approximate stable vector (to four decimal accuracy) for this process is

\[ \vec{S}_{\text{stable}} = \begin{bmatrix} 0.7586 \\ 0.2414 \end{bmatrix} \]

If \( k \) is the month at which this process stabilizes, then

(A) The correct answer is not here.

(B) ★ 10 \( \leq k \leq 12 \)

(C) 6 \( \leq k \leq 9 \)

(D) 13 \( \leq k \leq 14 \)

(E) 15 \( \leq k \leq 17 \)

Solution. The transition matrix and stable matrix of this process are

\[ T = \begin{bmatrix} 0.86 & 0.44 \\ 0.14 & 0.56 \end{bmatrix}, SM = \begin{bmatrix} 0.7586 & 0.7586 \\ 0.2414 & 0.2414 \end{bmatrix} \]

Also

\[ T^{10} = \begin{bmatrix} 0.7587 & 0.7585 \\ 0.2413 & 0.2415 \end{bmatrix} \quad \text{and} \quad T^{11} = \begin{bmatrix} 0.7586 & 0.7586 \\ 0.2414 & 0.2414 \end{bmatrix} \]

so \( k = 11 \).

You also received full credit if you compared the first four decimals without rounding. In such case your answer was \( k = 13 \) (13 \( \leq k \leq 14 \)). Note that the convention is to round first and then compare.
10. (5 points) Let \( \vec{u} = (-7, 1, 4) \) and let \( \vec{v} = (-1, 9, 5) \). For which value of \( k \) is the vector \((3k, 4k, 1)\) in the span of \( \vec{u} \) and \( \vec{v} \)?

(A) ★ \( k = 2 \)

(B) There are infinitely many values of \( k \) for which \((3k, 4k, 1)\) is in the span of \( \vec{u} \) and \( \vec{v} \).

(C) \( k = -1 \)

(D) There is no value of \( k \) for which \((3k, 4k, 1)\) is in the span of \( \vec{u} \) and \( \vec{v} \).

(E) \( k = 1 \)

Solution. The vector \((3k, 4k, 1)\) is in the span of \( \vec{u} \) and \( \vec{v} \) if and only if the system below is consistent.

\[
\begin{bmatrix}
-7 & -1 & 3k \\
1 & 9 & 4k \\
4 & 5 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & -k/2 \\
0 & 1 & k/2 \\
0 & 0 & -k/2 + 1
\end{bmatrix}
\]

The system is consistent only if \( 0 = -k/2 + 1 \). Solve for \( k \) to get \( k = 2 \). The vector \((3k, 4k, 1)\) is in the span of \( \vec{u} \) and \( \vec{v} \) only if \( k = 2 \).
11. (5 points) Consider the following statements:

(I) The span of two vectors in $\mathbb{R}^3$ is always a plane that passes through the origin.

(II) If $\mathbf{u}$ and $\mathbf{v}$ are two parallel vectors in $\mathbb{R}^n$, then $\text{span}\{\mathbf{u}, \mathbf{v}\} = \text{span}\{\mathbf{v}\}$.

(III) The span of two non-parallel vectors in $\mathbb{R}^2$ is always all of $\mathbb{R}^2$.

Which of the above statements are TRUE?

(A) All (I), (II), and (III) are true.

(B) Only (I) and (III) are true.

(C) ★ Only (II) and (III) are true.

(D) Only (I) is true.

(E) Only (III) is true.

Solution.
12. (5 points)

\[
A = \begin{bmatrix}
6 & 1 & -4 \\
-1 & 6 & 7 \\
11 & 8 & -1
\end{bmatrix}
\]

Out of the following vectors:

\( \vec{u} = (5, -1, 9) \)
\( \vec{v} = (4, -13, -5) \)
\( \vec{w} = (3, -3, 7) \)

which are in the column space of \( A \)?

(A) \( \star \) \( \vec{u} \) and \( \vec{v} \)

(B) \( \vec{u} \) and \( \vec{w} \)

(C) None of these vectors are in the column space of \( A \)

(D) All of these vectors are in the column space of \( A \)

(E) \( \vec{w} \) only

---

Solution.

\[
\begin{bmatrix}
6 & 1 & -4 \\
-1 & 6 & 7 \\
11 & 8 & -1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & -31/37 \\
0 & 1 & 38/37 \\
0 & 0 & 0
\end{bmatrix}
\]

The system with \( \vec{u} \) on the RHS is consistent, so \( \vec{u} \) is in the column space of \( A \).

The system with \( \vec{v} \) on the RHS is consistent, so \( \vec{v} \) is in the column space of \( A \).

The system with \( \vec{w} \) on the RHS is NOT consistent, so \( \vec{w} \) is NOT in the column space of \( A \).
13. (5 points) Let \( S = \{(4, -1, -1), (-2, 1, 3), (6, -2, -4)\} \).

Give a geometric description of the span of \( S \).

(A) Three points in \( \mathbb{R}^3 \)

(B) ★ A plane in \( \mathbb{R}^3 \)

(C) All of \( \mathbb{R}^3 \)

(D) A line in \( \mathbb{R}^3 \)

(E) A point in \( \mathbb{R}^3 \)

Solution.

\[
\begin{bmatrix}
4 & -2 & 6 \\
-1 & 1 & -2 \\
-1 & 3 & -4
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & -1 \\
0 & 0 & 0
\end{bmatrix}
\]

The RREF has 2 non-zero rows, so the span of \( S \) is 2-dimensional – a plane (in \( \mathbb{R}^3 \) because each vector has three coordinates).
14. (5 points) A paper company produces three kinds of paper at three different mills. The Huntington mill produces 4 tons of basic white paper, 8 tons of color paper, and 4 tons of stationary paper per 8-hour work day. The Charleston mill produces 3 tons of basic white paper, 2 tons of color paper, and 4 tons of stationary paper per 8-hour work day. The Parkersburg mill produces 8 tons of basic white paper, 12 tons of color paper, and 9 tons of stationary paper per 8-hour work day. The Parkersburg mill is expensive to run, so the company is exploring shut it down.

Is it possible that the other two mills can be run overtime to exactly make up for the daily output lost by shutting down the Parkersburg mill? If so, what is the new production schedule?

(A) Yes. To replace the Parkersburg mill, the Huntington mill must be run for a total of 20 hours per day and the Charleston mill for a total of 12 hours per day.

(B) No, because the production vector of Parkersburg mill cannot be expressed as a linear combination of the other two production vectors.

(C) Yes. To replace the Parkersburg mill, the Huntington mill must be run for a total of 16 hours per day and the Charleston mill for a total of 12 hours per day.

(D) No, because we would need to run one of the other mills for more than 24 hours per day.

(E) ✡ Yes. To replace the Parkersburg mill, the Huntington mill must be run for a total of 18 hours per day and the Charleston mill for a total of 16 hours per day.

Solution.

\[
\begin{bmatrix}
H & C & P
\end{bmatrix} =
\begin{bmatrix}
4 & 3 & 8 \\
8 & 2 & 12 \\
4 & 4 & 9
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 5/4 \\
0 & 1 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]

The system is consistent and the solution is \( c_1 = 5/4 \) and \( c_2 = 1 \). Thus, \( P = 5/4 \cdot H + 1 \cdot C \).

So, we need to run the Huntington mill for \( 8 + 5/4 \cdot 8 = 18 \) hours a day and the Charleston mill for \( 8 + 1 \cdot 8 = 16 \) hours a day.