Math 125 Fall 2017 Exam 1  
Monday, October 2, 2017

Name: ________________________________  NetID: ______________________@illinois.edu

Signature: ______________________________

Circle your lecture:

8am lecture  2pm lecture  3pm lecture

• No cells phones, i-Pods, MP3 players. Turn them off now. If you are seen these items in hand during the exam it will be considered cheating and you will be asked to leave.

• Print your proper name and NetID clearly at the top of this page, sign the exam below your name, and circle the section for which you are registered.

• There are 14 question in the exam, 2 of which are free response and 12 are multiple choice.

• In the free response part, you must show all your work to receive full credit.

• In the multiple choice part, you must circle your final choice with a regular pen (not a pencil!). Showing your work is NOT required for the multiple choice part.

• No questions will be permitted during the exam. If you are uncertain about something, put a small note about it on your exam paper and then answer the question as best you can with the information that you are given.

• You have 60 minutes to complete this exam.

• If you finish early, quietly and respectfully get up and hand in your exam.

• When time is up, put down your writing utensil, close your exam, and remain seated. Anyone seen continuing to write after time is called will have their exam marked and lose all points on the page they are writing on.

DO NOT OPEN EXAM UNTIL TOLD TO DO SO

Calculator approved? YES  NO
1. (20 points) ______________

2. (20 points) ______________

3. (5 points) ______________

4. (5 points) ______________

5. (5 points) ______________

6. (5 points) ______________

7. (5 points) ______________

8. (5 points) ______________

9. (5 points) ______________

10. (5 points) ______________

11. (5 points) ______________

12. (5 points) ______________

13. (5 points) ______________

14. (5 points) ______________

**TOTAL (100 points) __________**

92. B

93. A

94. A

95. E

96. D
1. (20 points) The company ‘Happy Garden’ builds garden sheds. A large shed requires 90 wall panels and 30 studs, and a small shed requires 30 wall panels and 20 studs. The company has available 270 wall panels and 120 studs. If they sell a large shed for $500 and a small shed for $300, how many of each type of building should they build to maximize their revenue?

Solution. No partial credit within each part. For example, you either obtain 2 points or 0 points for assigning the variable $x$.

Assign variables:
(2 points) $x = \text{the number of large sheds}$
(2 points) $y = \text{the number of small sheds}$

(2 points) Define the objective function: $R = 500x + 300y$

List all constraints:
(2 points) $90x + 30y \leq 270$
(2 points) $30x + 20y \leq 120$
(1 points) $x \geq 0, y \geq 0$

(8 points) Solve the linear program:
• Method 1: Draw the feasibility region. You should get a 4-sided polygon with vertices (0,0), (3,0), (2,3), (0,6). You find the point (2,3) by solving the system of equations $90x + 30y = 270$ and $30x + 20y = 120$. Since the feasibility region is a polygon, it is enough to evaluate the objective function $R$ at the corners. $R(0,0) = 0, R(3,0) = 1500, R(2,3) = 1900, (0,6) = 1800$. The maximum is 1900 and occurs at the point (2,3).

(Out of the 8 points, 3 points were for plotting the feasibility region, 2 points for finding the intersection point (2,3), and 3 points for evaluating the objective function at the corners.)

• Method 2: The initial simplex table is:
\[
\begin{bmatrix}
1 & -500 & -300 & 0 & 0 & 0 \\
0 & 90 & 30 & 1 & 0 & 270 \\
0 & 30 & 20 & 0 & 1 & 120 \\
\end{bmatrix}
\]

You first need to pivot on row 2, column 2. Then you need to pivot on row 3, column 3. The final simplex table is:
\[
\begin{bmatrix}
1 & 0 & 0 & 10/9 & 40/3 & 1900 \\
0 & 1 & 0 & 1/45 & -1/30 & 2 \\
0 & 0 & 1 & -1/30 & 1/10 & 3 \\
\end{bmatrix}
\]

The maximum is 1900. The basic feasible solution is (2, 3, 0, 0), and so the maximum occurs at $x = 2$ and $y = 3$.

(Out of the 8 points, 4 points were for the correct initial simplex table and 4 points for the correct final simplex table.)

(1 point) The maximum revenue of $1900$ dollars is attained by building 2 large sheds and 3 small sheds.
2. (20 points) You are organizing a dinner for your friends. You decide to make three types of tacos: regular, vegetarian, and supreme. One regular taco uses 2 oz of meat and 2 oz of beans. One vegetarian taco uses 0 oz of meat and 3 oz of beans. One supreme taco uses 4 oz of meat and 1 oz of beans. In your fridge, you have 14 oz of meat and 5 oz of beans. How many tacos of each type should you make if you intend to use all the meat and beans that you have available?

Solution. No partial credit within each part. For example, in the first part 'Assign variables' you either obtain 3 points or 0 points.

(3 points) Assign variables:

\[ x = \text{the number of regular tacos} \]
\[ y = \text{the number of vegetarian tacos} \]
\[ z = \text{the number of supreme tacos} \]

(4 points) The linear system to be solved is:

\[
\begin{align*}
2x + 4z &= 14 \text{ (meat)} \\
2x + 3y + z &= 5 \text{ (beans)}
\end{align*}
\]

(3 points) The RREF of the augmented matrix of this system is (you may use a calculator):

\[
\begin{bmatrix}
1 & 0 & 2 & 7 \\
0 & 1 & -1 & -3
\end{bmatrix}
\]

(4 points) The parametric solution of the system is:

\[
\{(7 - 2t, t - 3, t) \mid t \in \mathbb{R}\}
\]

(3 points) In order for all variables in this solution to have non-negative values, the parameter must lie in the interval:

\[
x \geq 0 \rightarrow 7 - 2t \geq 0 \rightarrow t \leq 3.5 \\
y \geq 0 \rightarrow t - 3 \geq 0 \rightarrow t \geq 3 \\
z \geq 0 \rightarrow t \geq 0 \\
\text{Therefore,} \\
3 \leq t \leq 3.5
\]

(3 points) How many tacos of each type should you make? Remember that you cannot make a fraction of a taco.

Since \( z = t \) must be an integer, we have \( t = 3 \). Then \( x = 7 - 2t = 1 \), \( y = t - 3 = 0 \), \( z = t = 3 \) (all integers). Therefore, the we make the following quantities of tacos:

regular tacos: 1  \quad \text{vegetarian tacos: } 0 \quad \text{supreme tacos: } 3
3. (5 points) For the simplex table below, what is the basic feasible solution?

\[
\begin{bmatrix}
1 & 3 & 0 & 2 & 0 & 9 & 0 & 22 \\
0 & 4 & 0 & 9 & 0 & 3 & 1 & 4 \\
0 & 1 & 1 & 5 & 0 & 7 & 0 & 6 \\
0 & 4 & 0 & 0 & 1 & 3 & 0 & 3
\end{bmatrix}
\]

(A) \( \star (x_1, x_2, x_3, s_1, s_2, s_3) = (0, 6, 0, 3, 0, 4) \)

(B) \((x_1, x_2, x_3, s_1, s_2, s_3) = (3, 0, 2, 0, 9, 0) \)

(C) \((x_1, x_2, x_3, s_1, s_2, s_3) = (4, 6, 3, 0, 0, 0) \)

(D) \((x_1, x_2, x_3, s_1, s_2, s_3) = (4, 0, 6, 0, 3, 0) \)

(E) \((x_1, x_2, x_3, s_1, s_2, s_3) = (0, 4, 0, 6, 0, 3) \)

**Solution.** Locate the columns which have the form \((0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\) and assign the correct RHS values. The column that corresponds to \(x_2\) has a 1 in the third row and the value in the third row of the RHS is 6, so \(x_2 = 6\). An analogous process gives \(s_1 = 3\) and \(s_3 = 4\). The remaining variables are 0.
4. (5 points) Reduced row echelon form (RREF) of a matrix is unique.

(A) ★ True
(B) False

Solution. The data for this question suggest that a large proportion of students did not understand what this question was asking. For this reason, everyone who answered this question (right or wrong) received 5 points.

You should know that no matter which elementary row operations you apply to a given matrix to bring it to RREF, you always get the same answer.
5. (5 points) Which of the following matrices are in row echelon form (REF)?

\[
M = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 2 & 0 & 3 \\
0 & 0 & 1 & 1 \\
1 & 1 & 3 & 1 \\
\end{bmatrix}, \quad
N = \begin{bmatrix}
1 & 2 & 3 & 7 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad
P = \begin{bmatrix}
1 & 0 & 3 & 2 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
\end{bmatrix},
\]

(A) All of them - M, N, and P

(B) Only N and P

(C) ★ Only N

(D) None of the matrices are in row echelon form

Solution. -
6. (5 points) The row echelon form of the augmented matrix of a linear system is shown here:

\[
\begin{bmatrix}
1 & 0 & 4 & a \\
0 & 1 & 1 & b \\
0 & 0 & 1 & c \\
\end{bmatrix}
\]

The linear system is in the variable \(x, y\) and \(z\), and \(a\) and \(b\) are real numbers. Express the solution for \(x, y\) and \(z\) in terms of \(a\) and \(b\).

(A) The correct solution is not here.
(B) \(\star x = a - 4c, \ y = b - c, \ z = c\)
(C) \(x = a + 4b, \ y = b + c, \ z = c\)
(D) \(x = 4a, \ y = b, \ z = 0\)

Solution. -
7. (5 points) What is the geometric shape of this solution set?

\[ \{(t + 3, 2t - 5, t) \mid t \in \mathbb{R}\} \]

(A) The empty set.

(B) A single point.

(C) The correct answer is not here.

(D) A plane.

(E) ★ A line.

Solution. We have one parameter \( t \), so this must be a line.
8. (5 points) Which of the following statements are true?

I. The row rank of
\[
\begin{bmatrix}
1 & 0 & 4 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\]
is 2.

II. The row rank of
\[
\begin{bmatrix}
1 & 0 & 4 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
\end{bmatrix}
\]
is 3.

III. The row rank of
\[
\begin{bmatrix}
1 & 0 & 4 \\
0 & 1 & 0 \\
0 & 0 & 2 \\
\end{bmatrix}
\]
is 3.

(A) ★ Only I and III are true.
(B) Only I is true.
(C) Only I and II are true.
(D) All of the above are true.

Solution. -
9. (5 points) Consider the following four statements about a homogeneous linear system:

(I) may have no solution
(II) may have just one solution
(III) may have exactly two solutions
(IV) may have infinitely many solutions

Which of these possibilities are true?

(A) Only I, II, and III
(B) Only I and IV
(C) All of the statements are true
(D) ★ Only II and IV
(E) Only I, II, and IV

Solution. Every homogeneous system has the trivial solution where all variables are zero (e.g. \((0,0,0)\)). If the rank of the augmented matrix equals the number of variables, then the system only has the trivial solution (so (IV) is correct). If the rank of the augmented matrix is strictly smaller than the number of variables, then the system only has infinitely many solutions (so (II) is correct). The rank of the augmented matrix can never be larger than the number of variables, so there is no other option.
10. (5 points) The picture below shows a feasibility region of a linear program in which we are to minimize some cost function \( C \). The cost function \( C \) is constant along each line of constancy \( \ell_1 \) and \( \ell_2 \). If \( C = 30 \) along the line \( \ell_1 \) and \( C = 20 \) along the line \( \ell_2 \), then the minimum cost occurs at the corner:

\[ (A) \ P_1 \]

\[ (B) \ P_3 \]

\[ (C) \ \star \ P_5 \]

\[ (D) \ P_4 \]

\[ (E) \ P_2 \]

**Solution.** Since \( C \) decreases when you move from \( \ell_1 \) to \( \ell_2 \), you must move the lines of constancy up and right to minimize \( C \). The last point of the feasibility region that you hit when moving the line up and right is \( P_5 \).
11. (5 points) Which of the following matrices are in reduced row echelon form (RREF)?

\[ M = \begin{bmatrix}
1 & 0 & 0 & 3 \\
0 & 1 & 1 & 2 \\
0 & 0 & 1 & 1
\end{bmatrix}, \quad N = \begin{bmatrix}
1 & 0 & 3 & 0 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad P = \begin{bmatrix}
1 & 2 & 2 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \]

(A) None of the matrices are in reduced row echelon form
(B) \(\star\) Only \(N\)
(C) Only \(M\) and \(N\)
(D) Only \(M\)

Solution. -
12. (5 points) Find the value for $k$ that makes the following system of equations INCONSISTENT.

\[
\begin{align*}
3x + 3z &= 9 \\
4x + y + 3z &= 1 \\
6x - y + k \cdot z &= 10
\end{align*}
\]

(A) ★ Only $k = 7$ makes the system inconsistent.

(B) Only $k = 19$ makes the system inconsistent.

(C) There is more than one value of $k$ that makes the system inconsistent.

(D) Only $k = 8$ makes the system inconsistent.

Solution.

\[
\begin{bmatrix}
3 & 0 & 3 & 9 \\
4 & 1 & 3 & 1 \\
6 & -1 & k & 10
\end{bmatrix}
\]

\[
\begin{array}{c}
R'_{1} = R_{1}/3 \\
R'_{2} = R_{2} - 4R_{1} \\
R'_{3} = R_{3} - 6R_{1}
\end{array}
\]

\[
\begin{bmatrix}
1 & 0 & 1 & 3 \\
0 & 1 & -1 & -11 \\
0 & -1 & k-6 & -8
\end{bmatrix}
\]

\[
\begin{array}{c}
R'_{3} = R_{3} + R_{2} \\
R'_{3} = R_{3} + R_{2}
\end{array}
\]

\[
\begin{bmatrix}
1 & 0 & 1 & 3 \\
1 & 0 & 1 & 3 \\
0 & 0 & k-7 & -19
\end{bmatrix}
\]

The last row is $[0 \ 0 \ 0 \ | \ \text{non-zero}]$ only if $k - 7 = 0$. So the only choice of $k$ that makes the system inconsistent is $k = 7$. 

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13. (5 points) Which of these graphs is the feasibility region for the following system of linear inequalities?

\[
\begin{align*}
5x + 4y &\leq 20 \\
2x + 5y &\leq 10 \\
x, y &\geq 0
\end{align*}
\]

- Graph I
- Graph II
- Graph III
- Graph IV

(A) ⭐ Graph I.
(B) Graph II.
(C) Graph IV.
(D) Graph III.

Solution.
14. (5 points) Consider the system of equations below:

\[
\begin{align*}
3x - 2y + 4z + 5w &= -17 \\
-x + 2y - 3z - 2w &= 21 \\
5x - 2y + 2z + 5w &= -7 \\
x, y, z, w, &\geq 0
\end{align*}
\]

\[
\begin{bmatrix}
3 & -2 & 4 & 5 & -17 \\
-1 & 2 & -3 & -2 & 21 \\
5 & -2 & 2 & 5 & -7
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 1 & 3 \\
0 & 1 & 0 & 1 & 9 \\
0 & 0 & 1 & 1 & -2
\end{bmatrix}
\]

Assuming you only want to consider non-negative solutions which are integers, how many solutions does this system have?

(A) Infinitely many solutions  
(B) 6 solutions  
(C) ★ No solutions  
(D) 4 solutions

**Solution.** Looking at the RREF, we have one free variable \( w \) (because the fourth column does not have a leading 1). Therefore, we need to introduce one parameter \( w = t \). The system of equations that corresponds to the RREF is

\[
\begin{align*}
x + t &= 3 \\
y + t &= 9 \\
z + t &= -2, \quad t \in \mathbb{R}
\end{align*}
\]

So the parametric solution is \( x = 3 - t, \ y = 9 - t, \ z = -2 - t \). Since all \( x, y, z, w \) need to be non-negative, we get the following constraints on \( t \):

\[
\begin{align*}
x \geq 0 \Rightarrow 3 - t \geq 0 \Rightarrow t \leq 3 \\
y \geq 0 \Rightarrow 9 - t \geq 0 \Rightarrow t \leq 9 \\
z \geq 0 \Rightarrow -2 - t \geq 0 \Rightarrow t \leq -2 \\
w \geq 0 \Rightarrow t \geq 0
\end{align*}
\]

But there is no \( t \) that satisfies both inequalities \( t \leq -2 \) and \( t \geq 0 \) (you cannot have a non-negative number smaller or equal than \(-2\)). So there is no \( t \) that satisfies all four constraints. This means that the system has no non-negative solution \( (x, y, z, w) \) at all. In other words, every solution to the above system will have at least one negative coordinate.

The analysis of the scantron data showed that this variant was harder for you than the other variants of this problem. To maintain similar average score on this question among variants, you were given 5 points for the correct answer and 2 points for an incorrect answer.