60 points possible

1. No Hats or dark sunglasses. All hats are to be removed.

2. All book bags are to be closed and placed in a way that makes them inaccessible. Do not reach into your bag for anything during the exam. If you need extra pencils, pull them out now.

3. No cell phones. Turn them off now. If you are seen with a cell phone in hand during the exam, it will be construed as cheating and you will be asked to leave. This includes using it as a time-piece.

4. No music systems – IPods, MP3 players, etc.; same rules as with cell phones.

5. On this exam, if the problem is able to be done using the calculator or does not explicitly say not to use a calculator, you may use your calculator to solve.

6. Absolutely no sharing of calculators. Both parties will be considered equal party in the cheating and asked to leave.

7. To receive ANY credit on problems that require row operations by hand, you must clearly state and show all the steps.

8. If you have a question, raise your hand and a proctor will come to you. Once you stand up, you are done with the exam. If you have to use the facilities, do so now. You will not be permitted to leave the room and return during the exam.

9. Every exam is worth a total of 60 points. Check to see that you have all of the pages. Including the cover sheet, each exam has 8 pages.

10. Be sure to print your proper name clearly and then write down the lecture section (i.e. B (9AM), X (12PM) or F(2PM)) for which you are registered.

11. If you finish early, quietly and respectfully get up and hand in your exam. You need to show your student ID when you hand in the exam. (Drivers license, passport, etc. will work also.) No exam will be accepted without ID.

12. When time is up, you will be instructed to put down your writing utensil, close the exam and remain seated. Anyone seen continuing to write after this announcement will have their exam marked and lose all points on the page they are writing on. We will come and collect the exams from you. Have your ID ready.

13. Good luck. You have ???? minutes to complete the exam.
1. Let $A$, $B$ and $C$ be $n \times n$ matrices.

(a) (5pts) If the determinant of $A$ is $-4$, what is the reduced row echelon form of $A$? Briefly explain your answer.

\[ \text{ref} (A) = \begin{bmatrix} 1 \\ \end{bmatrix} \text{ the identity matrix} \]

\[ \text{det} (A) \neq 0 \iff A \text{ is invertible} \iff \text{ref} (A) = I \]

(b) (5pts) If $A\vec{x} - B\vec{x} = \vec{0}$, under what conditions does this system have a unique solution for $\vec{x}$? Justify your answer.

The system has a unique solution

\[ \text{iff } (A - B) \text{ is invertible.} \]

\[ \vec{x} = (A - B)^{-1} \vec{0} \]
2. (3pts) Let $A$ and $B$ be $n \times n$ matrices. If the $\det A = 3$ and $\det B = -2$, determine $\det (AB)$. Show work.

\[
\det(AB) = \det(A) \det(B) \\
= 3 \cdot (-2) \\
= -6
\]

(b) (7pts) Let $A$ be invertible. (That is $A^{-1}$ exists). Show that $\det A^{-1} = \frac{1}{\det A}$.

\[
AA^{-1} = I \\
\det(AA^{-1}) = \det(I) \\
\det(A) \det(A^{-1}) = 1 \\
\det(A^{-1}) = \frac{1}{\det(A)}
\]
3. A survey studying the social networking usage of college students found that 60% percent of students use Facebook, 25% use MySpace, and 15% use Friendster. During any given month there is a 5% probability that a Facebook user will switch to using MySpace and a 2% probability that a Facebook user will switch to Friendster. For MySpace users there is a 5% probability of switching to Facebook and a 10% probability of switching to Friendster. For Friendster users there is a 20% probability of switching to Facebook and a 15% probability of switching to MySpace.

(a) (2pts) Construct the transition matrix defined by this model. Clearly label the matrix.

\[
T = \begin{bmatrix} 0.93 & 0.05 & 0.02 \\
0.05 & 0.85 & 0.15 \\
0.02 & 0.10 & 0.65 \end{bmatrix}
\]

(b) (3pts) What percentage of people will be Friendster users after 4 months? (Round your answer to 4 decimal places.)

\[
S_0 = \begin{bmatrix} 0.60 \\
0.25 \\
0.15 \end{bmatrix} \quad S_4 = T^4 S_0 = \begin{bmatrix} 0.5924 \\
0.2885 \\
0.1191 \end{bmatrix}
\]

11.91% of students will be Friendster users.

(c) (5pts) Determine the exact stable vector for this Markov process. (Make sure the stable vector is in fractions.) In the long run, what percentage of the population will be Facebook users?

\[
T - I = \begin{bmatrix} -0.07 & 0.05 & 0.2 \\
0.05 & -0.15 & 0.15 \\
0 & -0.1 & 0.35 \end{bmatrix} \quad O = \begin{bmatrix} 0 \\
0 \\
1 \end{bmatrix}
\]

\[
\hat{S} = \left( \begin{bmatrix} -0.07 & 0.05 & 0.2 \\
0.05 & -0.15 & 0.15 \\
0 & -0.1 & 0.35 \end{bmatrix}^{-1} \right) \begin{bmatrix} 0 \\
0 \\
1 \end{bmatrix} = \begin{bmatrix} 25/44 \\
41/132 \\
4/33 \end{bmatrix}
\]

In the long run, \(\frac{25}{44}\) of the students will be Facebook users.
4. Los Angeles has three industries: drugs and prostitution, Hollywood (TV and film), and a Police force. Let $1 of drugs and prostitution require $0.20 in drugs and prostitution, $0.10 in Hollywood, and $0.15 in Police force. Let $1 of Hollywood production use $0.80 in drugs and prostitution, $0.15 in Hollywood and $0.10 Police force, while a $1 in Police force takes $0.20 in drugs and prostitution and $0.10 in Hollywood.

(a) (3pts) Construct the consumption matrix for this model. Clearly label the matrix.

\[
\mathbf{C} = \begin{bmatrix}
0.20 & 0.20 & 0.10 & \text{usage of D & P} \\
0.10 & 0.15 & 0.15 & \text{usage of HW} \\
0.15 & 0.10 & 0 & \text{usage of PF}
\end{bmatrix}
\]

(b) (2pts) Define what it means for an economy to be productive.

An economy is productive if it can meet any external demand. Given any demand \( \mathbf{D} \), there is a production vector \( \mathbf{x} \) with non-negative entries such that \( (\mathbf{I} - \mathbf{C})^{-1}\mathbf{x} = \mathbf{D} \).

(c) (2pts) Clearly explain why this economy is a productive economy.

\[
(\mathbf{I} - \mathbf{C})^{-1} = \begin{bmatrix}
\frac{328}{221} & \frac{328}{221} & \frac{100}{221} \\
\frac{46}{221} & \frac{305}{221} & \frac{10}{221} \\
\frac{55}{221} & \frac{50}{221} & \frac{240}{221}
\end{bmatrix}
\]

The economy is productive because \( (\mathbf{I} - \mathbf{C})^{-1} \) has all non-negative entries.

(d) (3pts) Find the production schedule if the demand is for $500,000,000 in drugs and prostitution, $1,500,000,000 in Hollywood (TV and film), and $100,000,000 in Police force.

\[
\mathbf{D} = \begin{bmatrix}
5 \\
15 \\
1
\end{bmatrix}
\]

\[
\mathbf{x} = (\mathbf{I} - \mathbf{C})^{-1} \mathbf{D} = \begin{bmatrix}
30.3167 \\
3167 \\
7.7602
\end{bmatrix}
\]

They must produce $3.03167 billion in Drugs and Prostitution,

$2.31267 billion in Hollywood,

and $776.02 million in Police Force.
5. Let \( V \) be a real vector space with operations \( \oplus \) and \( \odot \).

(a) (3pts) Define the additive identity element for the vector space \( V \).

The additive identity is a vector \( \overrightarrow{0} \) such that

\[
\overrightarrow{v} \oplus \overrightarrow{0} = \overrightarrow{v}
\]

for all vectors \( \overrightarrow{v} \) in \( V \).

(b) (4pts) State three other properties of the definition of a vector space.

1) closure under addition
2) addition is commutative
3) addition is associative
4) additive inverse exist
5) closure under scalar multiplication
6) closure under scalar multiplication
7) \( c \odot (\overrightarrow{u} \oplus \overrightarrow{v}) = (c \odot \overrightarrow{u}) \oplus (c \odot \overrightarrow{v}) \)
8) \( (c \odot \overrightarrow{u}) \oplus (d \odot \overrightarrow{u}) = (c + d) \odot \overrightarrow{u} \)
9) \( c \odot (l \odot \overrightarrow{u}) = (c l) \odot \overrightarrow{u} \)
10) \( 1 \odot \overrightarrow{u} = \overrightarrow{u} \)

(c) (3pts) Let \( V \) be the set of all non-negative real numbers (\( V = [0, \infty) \)). \( (V, \oplus, \odot) \) is a vector space when vector addition is designed to be

\[
a \oplus b = ab
\]

for any two numbers \( a \) and \( b \) in \( V \) and vector scalar multiplication is defined to be

\[
r \odot a = a^r
\]

for any real number \( r \). Determine the zero vector for this vector space. Clearly justify your answer.

The zero vector is the number \( 1 \).

Let \( r \) be any positive number (or 0).

\[
r \oplus 1 = r \cdot 1 = r
\]

1. satisfies the definition of \( \overrightarrow{0} \) so

\[
\overrightarrow{0} = 1.
\]
6. (5pts) Consider the set $W$ consisting of all $2 \times 3$ matrices of the form \[
\begin{bmatrix}
  a & b & c \\
  d & e & f
\end{bmatrix}
\] where \(a, b, c, d, e\) and \(f\) are non-negative real numbers. Determine if the set $W$ is a subspace of $M_{2\times3}$.

This set is not a subspace of $M_{2\times3}$.

It is not closed under scalar multiplication:

\[
(-1) \begin{bmatrix}
  0 & 1 & 2 \\
  3 & 4 & 5
\end{bmatrix} = \begin{bmatrix}
  0 & -1 & -2 \\
  -3 & -4 & -5
\end{bmatrix}
\]

The result is not in the set.
7.

(a) (3pts) Define what it means for a set of vectors to be linearly independent.

A set \( S = \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \} \) is linearly independent if whenever \( c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_k \vec{v}_k = \vec{0} \) we must have \( c_1 = 0, c_2 = 0, \ldots, c_k = 0 \).

(Or: None of the vectors can be written as a linear combination of the others)

(b) (2pts) Determine if the vectors \((4, 1, 5), (6, 8, -2), (9, -9, 3)\) and \((1, -7, 4)\) are linearly independent. Justify your answer.

\[
\begin{bmatrix}
4 & 6 & 9 & 1 & 0 \\
1 & 8 & -9 & -7 & 0 \\
5 & -2 & 3 & 4 & 0
\end{bmatrix}
\xrightarrow{\text{ref}}
\begin{bmatrix}
1 & 0 & 0 & \frac{41}{107} & 0 \\
0 & 1 & 0 & \frac{131}{107} & 0 \\
0 & 0 & 1 & \frac{34}{107} & 0
\end{bmatrix}
\]

The system has nontrivial solutions so these vectors are linearly dependent.