The following linear program was set up by a New Jersey based employee of Toyota. She is trying deliver Priuses to the ecologically-sound employees of Rutgers University.

\[ x = \text{number of Priuses to be shipped from Newark to New Brunswick} \]
\[ y = \text{number of Priuses to be shipped from Trenton to New Brunswick} \]

The variables \( x \) and \( y \) must satisfy the following constraints:

\[ x + y \leq 15 \]
\[ x + y \geq 12 \]
\[ 0 \leq x \leq 10 \]
\[ 0 \leq y \leq 12 \]

The cost function is given by \( C(x, y) = 1260 - 10x - 15y \).

(a) (6pts) Graph the feasibility region and find the corner points.

**Solution**: Begin by graphing the line \( x + y = 15 \), which has the intercepts. (Graphically, this is the red-line.) Using the point \((0, 0)\) as a test point, we have the question “Is \((0) + (0) = 0 \leq 15\)” Yes, it is. Therefore, the half-plane defined by \( x + y \leq 15 \) is the region possessing and below the red line.

Similarly, graphing \( x + y = 12 \) we draw the blue line and find that the half plane defined by the in \( x + y \geq 12 \) is the region including and above the line. Combining the two half planes, we find the following unbounded feasibility region:

We also need to graph the lines \( x = 10 \) and \( y = 12 \). Clearly \( x \leq 10 \) is the region to the left of the vertical line and \( y \leq 12 \) is the region below the horizontal line.

The corner points are \((0, 12), (3, 12), (10, 5)\) and \((10, 2)\).
(b) (4pts) Solve the linear program.

**Solution**: Since we already have the corner points of the polygonal feasibility region, the easiest way to solve the linear program is to evaluate $C$ at the corner points.

\[
\begin{align*}
C(0, 12) &\rightarrow C = 1260 - 10(0) - 15(12) = 1080 \\
C(3, 12) &\rightarrow C = 1260 - 10(3) - 15(12) = 1050 \\
C(10, 5) &\rightarrow C = 1260 - 10(10) - 15(5) = 1085 \\
C(10, 2) &\rightarrow C = 1260 - 10(10) - 15(2) = 1130 \\
\end{align*}
\]

*The conclusion*: The minimum cost is $1050 and can be incurred by shipping to New Brunswick 3 Priuses from Newark and 12 from Trenton.
2. (10pts) A small shoe manufacturer makes two styles of shoes: oxfords and loafers. Two machines are used in the process: a cutting machine and a sewing machine. Each type of shoe requires 1/4 hour per pair on the cutting machine. The oxfords require 1/6 hour of sewing per pair and loafers require 1/3 hour of sewing per pair. Because the manufacturer can hire only one operator for each machine, each process is available for just 8 hours per day. The profit from the sale of a pair of oxfords is $15 and is $20 for each pair of loafers. Set up but do not solve a linear program that will determine how many pairs of shoes should be produced to maximize profit.

Solution: Let \( x \) = the number of pairs of oxfords made and \( y \) = the number of pairs of loafers made.

The objective here is to maximize the profit. The profit function is \( P = 15x + 20y \).

Each machine can run for at most 8 hours a day, for the cutting machine, we get the inequality \( \frac{1}{4}x + \frac{1}{4}y \leq 8 \).

Similarly, for the sewing machines, we get the inequalities \( \frac{1}{6}x + \frac{1}{3}y \leq 8 \).

We also know that our variables need to be non-negative. (It doesn’t make sense to run a machine a negative number of hours.) So we get the final inequalities, \( x \geq 0 \) and \( y \geq 0 \).

The above discussion accurately describes the linear program. One could choose to represent this in its mathematical form:

\[
\begin{align*}
\text{maximize} & \quad P = 15x + 20y \\
\text{subject to} & \quad \frac{1}{4}x + \frac{1}{4}y \leq 8 \\
& \quad \frac{1}{6}x + \frac{1}{3}y \leq 8 \\
& \quad x \geq 0 \\
& \quad y \geq 0
\end{align*}
\]

(Note: You did not have to order the linear program like this for credit.)
3. (5pts) Determine the best description of the following matrices: row echelon form, reduced row echelon form or neither.

(a) \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Solution: For row echelon form, any rows of all zeros must be on the bottom. This is neither in ref or rref.

(b) \[
\begin{bmatrix}
1 & -1 & 5 \\
0 & 1 & -3 \\
0 & 0 & 0
\end{bmatrix}
\]

Solution: Every row’s first non-zero number is a 1. Below every leading 1 is a zero. The leading 1’s move to the left as you go down rows. This matrix satisfies the definition of row echelon form.

(c) \[
\begin{bmatrix}
0 & 1 & 0 & 4 \\
0 & 0 & 1 & 9 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Solution: This matrix satisfies the definition of row echelon. Additionally, above every leading 1 is a 0. This matrix is in reduced row echelon form.
4. (5pts) Consider the augmented matrix
\[
\begin{bmatrix}
1 & -1 & 5 & 19 \\
0 & 1 & -3 & -11 \\
0 & 0 & 1 & 5
\end{bmatrix}
\]
Perform back addition by hand and solve the corresponding system of equations.

Solution:

\[
\text{augmented}
\begin{bmatrix}
1 & -1 & 5 & 19 \\
0 & 1 & -3 & -11 \\
0 & 0 & 1 & 5
\end{bmatrix}
\]

\[
R'_1 = R_1 - 5R_3
\begin{bmatrix}
1 & -1 & 0 & -6 \\
0 & 1 & 0 & 4 \\
0 & 0 & 1 & 5
\end{bmatrix}
\]

\[
R'_2 = R_2 + 3R_1
\begin{bmatrix}
1 & 0 & 0 & -2 \\
0 & 1 & 0 & 4 \\
0 & 0 & 1 & 5
\end{bmatrix}
\]

The solution point is \((-2, 4, 5)\).
5.

(a) (3pts) Define the row rank of a matrix.

Solution: The row rank of a matrix is the number of non-zero rows the matrix has after it is in row echelon form.

(b) (2pts) Determine the row rank of the matrix

\[
\begin{bmatrix}
1 & 2 & 1 & 1 \\
2 & -1 & 1 & 2 \\
4 & 3 & 3 & 4 \\
3 & 1 & 2 & 3 \\
\end{bmatrix}
\]

Solution: The reduced row echelon form of this matrix is

\[
\begin{bmatrix}
1 & 0 & 3/5 & 1 \\
0 & 1 & 1/5 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]. Two nonzero rows tell us that the row rank is 2.
6. The following augmented matrices correspond to systems of linear equations. For each, determine the following:

1. the number of equations in the original system
2. the number of variables in the original system
3. the row rank of the augmented matrix
4. the number of parameters in the solution set
5. the number of the points in the solution set and the geometric interpretation of the solution set

(a) (5pts) \[
\begin{bmatrix}
1 & 5 & 0 & -6 & | & 0 \\
0 & 0 & 1 & -7 & | & 0 \\
0 & 0 & 0 & 0 & | & 1 \\
0 & 0 & 0 & 0 & | & 0 \\
\end{bmatrix}
\]

Solution:

1. 4 rows means 4 equations
2. 4 columns in the coefficient matrix mean 4 variables
3. 3 nonzero rows
4. The system is inconsistent. No parameters.
5. No solution points or \(\emptyset\).
(b) \[ \begin{bmatrix}
1 & 5 & 0 & 0 & 0 & -4 \\
0 & 0 & 1 & 0 & 0 & 5 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \]

1. 5 rows means 5 equations

2. 5 columns in the coefficient matrix mean 5 variables

3. 4 nonzero rows

4. The second variable is a free variable: 1 parameter

5. There are an infinite number of solutions in a parametric solution. One parameter solutions represent a line.
7. (10pts) Find the solution set for the following system of equations.

\[
\begin{align*}
    x_1 - 2x_2 - 8x_3 + 15x_4 + x_5 &= -8 \\
    3x_1 - 5x_2 - 21x_3 + 41x_4 + 3x_5 &= -19 \\
    -2x_1 + 7x_2 + 25x_3 - 42x_4 - 2x_5 &= 31
\end{align*}
\]

**Solution:** following augmented matrix:

\[
\begin{bmatrix}
    1 & -2 & -8 & 15 & 1 & | & -8 \\
    3 & -5 & -21 & 41 & 3 & | & -19 \\
    -2 & 7 & 25 & -42 & -2 & | & 31
\end{bmatrix}
\]

Using the calculator, we see that the reduced row echelon form of this matrix is

\[
\begin{bmatrix}
    1 & 0 & -2 & 7 & 1 & | & 2 \\
    0 & 1 & 3 & -4 & 0 & | & 5 \\
    0 & 0 & 0 & 0 & 0 & | & 0
\end{bmatrix}
\]

First, we notice that the system is consistent since the row ranks of the coefficient matrix and augmented matrix are both 2. Since the third variable column (the \(x_3\)-column) doesn’t have a leading 1 in it, we define \(x_3 = r\). Similarly, no leading 1 in the fourth and fifth column yields \(x_4 = s\) and \(x_5 = t\). The above matrix then yields the resulting system of equations:

\[
\begin{align*}
    x_1 - 2r + 7s + t &= 2 \\
    x_2 + 3r - 4s &= 5
\end{align*}
\]

This results in the solution set

\[
\{(2 + 2r - 7s - t, 5 - 3r + 4s, r, s, t) \text{ for all } r, s, \text{ and } t\}.
\]
8.
(a) (3pts) Define a homogeneous system of equations.

Solution: A system of equation is homogeneous if every equation in the system equals zero.

(b) (2pts) Explain why a homogeneous system is always consistent.

Solution: A system is consistent if it has one or more solutions. Since the zero solution (or the trivial solution) always solves the homogeneous system, the system must be consistent.
1. (10pts) A land developer plans to build a new subdivision with colonial-style and ranch-style homes. The colonial-style home requires $30,000 of capital and produces a profit of $4,000 when sold. The ranch-style home requires $40,000 of capital and produces $8,000 profit. The developer has $3.6 million of capital to invest and the land can be subdivided into at most 100 building sites. Set up but do not solve a linear program that will determine how many of each type of home to produce in order to maximize profit.

Solution: Let \( x \) = the number of colonial-style homes made and \( y \) = the number of ranch-style homes made.

The objective here is to maximize the profit. The profit function is \( P = 4,000x + 8,000y \).

The developer has only $3.6 million of capital. Knowing the cost to manufacture each home, we get the inequality \( 3,0000x + 4,0000y \leq 3,600,000 \).

Similarly, the 100 plots of land yield the inequalities \( x + y \leq 100 \).

We also know that our variables need to be non-negative. (It doesn’t make sense to manufacture a negative number of homes.) So we get the final inequalities, \( x \geq 0 \) and \( y \geq 0 \).

The above discussion accurately describes the linear program. One could choose to represent this in its mathematical form:

\[
\text{maximize} \quad P = 4000x + 8000y \\
\text{subject to} \quad 3,0000x + 4,0000y \leq 3,600,000 \\
\quad x + y \leq 100 \\
\quad x \geq 0 \\
\quad y \geq 0
\]
(Note: You did not have to order the linear program like this for credit.)
2. Due to the Super Bowl sales, the local Best Buy sold out of flat-panel TVs. An employee is ordering more from the corporate warehouses in Chicago and Indianapolis. Being a conscientious employee, he has set-up the following linear program.

\[ x = \text{number of TVs shipped from Chicago to Champaign} \]
\[ y = \text{number of TVs shipped from Indy to Champaign} \]

The variables \( x \) and \( y \) must satisfy the following constraints:

\[ x + y \leq 24 \]
\[ x + y \geq 16 \]
\[ 0 \leq x \leq 15 \]
\[ 0 \leq y \leq 19 \]

The cost function is given by \( C(x, y) = 1945 - 10x - 15y \).

(a) (6pts) Graph the feasibility region and find the corner points.

**Solution:** Begin by graphing the line \( x + y = 24 \), which has the intercepts. (Graphically, this is the red-line.) Using the point \((0, 0)\) as a test point, we have the question “Is \((0) + (0) = 0 \leq 24?\)” Yes, it is. Therefore, the half-plane defined by \( x + y \leq 24 \) is the region possessing and below the red line.

Similarly, graphing \( x + y = 16 \) we draw the blue line and find that the half plane defined by the in \( x + y \geq 16 \) is the region including and above the line. Combining the two half planes, we find the following unbounded feasibility region:

We also need to graph the lines \( x = 15 \) and \( y = 19 \). Clearly \( x \leq 15 \) is the region to the left of the vertical line and \( y \leq 19 \) is the region below the horizontal line.

By inspection, the corner points are \((0, 16), (0, 19), (5, 19), (15, 9) \) and \((15, 1)\).
(b) (4pts) Solve the linear program.

**Solution:** Since we already have the corner points of the polygonal feasibility region, the easiest way to solve the linear program is to evaluate $C$ at the corner points.

\[
\begin{align*}
C(0, 16) & \quad \rightarrow \quad C = 1945 - 10(0) - 15(16) = 1705 \\
C(0, 19) & \quad \rightarrow \quad C = 1945 - 10(0) - 15(19) = 1665 \\
C(5, 19) & \quad \rightarrow \quad C = 1945 - 10(5) - 15(19) = 1610 \\
C(15, 9) & \quad \rightarrow \quad C = 1945 - 10(15) - 15(9) = 1660 \\
C(15, 2) & \quad \rightarrow \quad C = 1945 - 10(15) - 15(2) = 1780
\end{align*}
\]

*The conclusion:* The minimum cost is $1610 and can be incurred by shipping 5 TV’s from Chicago to Champaign and 19 TVs from Indianapolis to Champaign.
3. Consider the augmented matrix
\[
\begin{bmatrix}
1 & -1 & 5 & 19 \\
0 & 1 & -3 & -11 \\
0 & 0 & 1 & 5 \\
\end{bmatrix}
\].

(a) (1pts) Determine the system of linear equations that corresponds to this augmented matrix.

Solution:

\[
\begin{align*}
\begin{aligned}
x - y + 5z &= 19 \\
y - 3z &= -11 \\
z &= 5
\end{aligned}
\end{align*}
\]

(b) (4pts) Perform \textit{back substitution} to solve the system of equations.

Solution: Using the value of \(z\) to solve for \(y\):

\[
\begin{align*}
\begin{aligned}
y - 3(5) &= -11 \\
y &= 4
\end{aligned}
\end{align*}
\]

Now using the values of \(y\) and \(z\) to solve for \(x\):

\[
\begin{align*}
\begin{aligned}
x + (3a - 6) - (2a - 3) &= 0 \\
x - (4) + 5(5) &= 19 \\
x &= -2
\end{aligned}
\end{align*}
\]

The solution point is \((-2, 4, 5)\).
4. 

(a) (4pts) Define what it means for a linear system to be consistent.

*Solution*: A system is consistent if it has one or more solution.

(b) (6pts) Consider the system

\[
\begin{align*}
  x + 2y + z &= 6 \\
-2x - 3y - 5z &= -7 \\
 5x + 7y + 14z &= k
\end{align*}
\]

Are there any values of \(k\) for which the system will be consistent? Clearly justify your answer.

*Solution:*

\[
\begin{bmatrix}
  1 & 2 & 1 & 6 \\
  -2 & -3 & -5 & -7 \\
  5 & 7 & 14 & k
\end{bmatrix}
\]

\[
R'_2 = R_2 + 2R_1 \quad \begin{bmatrix}
  1 & 2 & 1 & 6 \\
  0 & 1 & -3 & 5 \\
  0 & -3 & 9 & k - 30
\end{bmatrix}
\]

\[
R'_3 = R_3 - 5R_1 \quad \begin{bmatrix}
  1 & 2 & 1 & 6 \\
  0 & 1 & -3 & 5 \\
  0 & 0 & 0 & k - 15
\end{bmatrix}
\]

The row rank of the coefficient matrix is 2. In order for the system to be consistent, the augmented matrix must be of rank 2 as well. This requires that \(k - 15 = 0\) or \(k = 15\).
5. The following augmented matrices correspond to systems of linear equations. For each, determine the following:

1. the number of equations in the original system
2. the number of variables in the original system
3. the row rank of the augmented matrix
4. the number of parameters in the solution set
5. the number of the points in the solution set and the geometric interpretation of the solution set

(a) (5pts)

\[
\begin{bmatrix}
1 & 5 & 2 & -6 & 9 & 0 \\
0 & 0 & 1 & -7 & 4 & -8 \\
0 & 0 & 0 & 0 & 1 & -4 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

1. 5 rows means 5 equations
2. 5 columns in the coefficient matrix mean 5 variables
3. 3 nonzero rows
4. The second and the fourth variables are free variable: 2 parameters
5. There are an infinite number of solutions in a parametric solution. Two parameter solutions represent planes.
(b) (5pts) \[
\begin{bmatrix}
1 & 2 & 5 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
-4 \\
5 \\
1
\end{bmatrix}
\]

1. 3 rows means 3 equations

2. 3 columns in the coefficient matrix mean 3 variables

3. 3 nonzero rows

4. The system is inconsistent. No parameters.

5. No solution points or ∅.
6. (a) (2pts) Write the equation of the plane \(4x + 5y + 6z = 120\) in intercept form.

**Solution:**

\[
\frac{x}{30} + \frac{y}{24} + \frac{z}{20} = 1.
\]

(b) (3pts) Sketch the first-octant portion of the plane by drawing lines between the intercepts.

**Solution:** Recall the advantage of the equation being in intercept form is that it is significantly easier to find the intercepts of the plane. The \(x\)-intercept is \((30, 0, 0)\), the \(y\)-intercept is \((0, 24, 0)\) and the \(z\)-intercept is \((0, 0, 20)\). Plotting these points on the axis in the first-octant and connecting the line segments between the intercepts, we find the following portion of the plane:
7. (10pts) Find the solution set for the following system of equations.

\[
\begin{align*}
 x_1 + 3x_2 + 5x_3 + 33x_4 - 10x_5 &= 49 \\
-x_1 - 3x_2 - 4x_3 - 26x_4 + 7x_5 &= -40 \\
2x_1 + 6x_2 + 8x_3 + 52x_4 - 14x_5 &= 80
\end{align*}
\]

**Solution:** following augmented matrix:

\[
\begin{bmatrix}
 1 & 3 & 5 & 33 & -10 & | & 49 \\
-1 & -3 & -4 & -26 & 7 & | & -40 \\
2 & 6 & 8 & 52 & -14 & | & 80
\end{bmatrix}
\]

Using the calculator, we see that the reduced row echelon form of this matrix is

\[
\begin{bmatrix}
 1 & 3 & 0 & -2 & 5 & | & 4 \\
0 & 0 & 1 & 7 & -3 & | & 9 \\
0 & 0 & 0 & 0 & 0 & | & 0
\end{bmatrix}
\]

First, we notice that the system is consistent since the row ranks of the coefficient matrix and augmented matrix are both 2. Since the third variable column (the \(x_2\)-column) doesn’t have a leading 1 in it, we define \(x_2 = r\). Similarly, no leading 1 in the fourth and fifth column yields \(x_4 = s\) and \(x_5 = t\). The above matrix then yields the resulting system of equations:

\[
\begin{align*}
 x_1 + 3r - 2s + 5t &= 4 \\
 x_3 + 7s - 3t &= 9
\end{align*}
\]

This results in the solution set

\[
\{(4 - 3r + 2s - 5t, r, 9 - 7s + 3t, s, t) \text{ for all } r, s, \text{ and } t\}.
\]