Part I - Free Response

1. (10 points)

(a) (5 points) Find all conditions on \( a \) and \( b \) that ensure that the matrix \( A \) is invertible. Explain your steps.

\[
A = \begin{bmatrix}
1 & 1 & 2 & 3 \\
-6 & 0 & 5 & -4 \\
a & b & 0 & 0 \\
12 & 9 & 11 & 10
\end{bmatrix}
\]

Invertible when \( \det(A) \neq 0 \)

Expand across row 3

\[
\det(A) = a M_{31} + b M_{32} + 0 = 0 \quad (1)
\]

\[
= a \begin{vmatrix}
1 & 2 & 3 \\
0 & 5 & 4 \\
9 & 11 & 0
\end{vmatrix} - b \begin{vmatrix}
1 & 2 & 3 \\
-6 & 0 & 4 \\
12 & 11 & 0
\end{vmatrix} \quad (2)
\]

\[
= a (-113) - b (-260) \\
= -113a + 260b
\]

Matrix \( A \) is not invertible when \( -113a + 260b = 0 \) \( \neq 0 \) \( (2) \)

(b) (5 points) \( A \) is a \( 2 \times 2 \) matrix that is invertible and

\[
B = \begin{bmatrix}
4 & 2 \\
3 & 4
\end{bmatrix}
\]

Find the determinant of \( BAB^T A^{-1} \). Explain your reasoning.

\[
\det(B) = 16 - 6 = 10 \quad (1)
\]

\[
\det(BAB^T A^{-1}) = \det(B) \det(A) \det(B^T) \det(A^{-1}) \\
= 10 \cdot \det(A) \cdot 10 \cdot \frac{1}{\det(A)} \quad (1)
\]

\[
= 100 \quad (1)
\]
2. (20 points) The Coke Company is interested in predicting the long run production schedule for three of its products. A survey of Coke drinkers shows that 30% of Coke Classic drinkers switch to Diet Coke and 30% switch to Coke Zero in one year. 20% of Diet Coke drinkers switch to Coke Classic and 20% to Coke Zero in a year. And 30% of Coke Zero drinkers switch to Coke Classic and 30% to Diet Coke in the same year. At the time of the survey, 25% drink Coke Classic, 35% drink Diet Coke and 40% drink Coke Zero.

(a) (3 points) What is the transition matrix for this problem (with labels)?

\[
T = \begin{bmatrix}
0.4 & 0.2 & 0.3 \\
0.3 & 0.6 & 0.3 \\
0.3 & 0.2 & 0.4 \\
\end{bmatrix}
\]

(b) (4 points) What will the distribution be after 2 years?

\[
S_0 = \begin{bmatrix}
0.25 \\
0.35 \\
0.40 \\
\end{bmatrix}
\]

\[
S_2 = T^2 S_0
\]

(c) (10 points) Find the exact stable vector and interpret what it tells you.

\[
(T - I)S = 0
\]

(d) (3 points) How many years does it take for this process to stabilize (to 4 decimals)?

After 9 years
3. An economy has three main sectors: labor, energy and manufacturing. One dollar worth of labor sector production uses $0.20 of labor, $0.40 of energy, and $0.00 of manufacturing. One dollar worth of energy sector production uses $0.00 of labor, $0.20 of energy, and $0.40 of manufacturing. One dollar worth of manufacturing sector production uses $0.10 of labor, $0.10 of energy, and $0.30 of manufacturing. Outside demand for the products of these sectors are $20 billion for labor, $10 billion for energy, and $30 billion for manufacturing.

(a) (5 points) What are the consumption matrix and demand vector of this process (with labels)?

\[
C = \begin{bmatrix}
1.2 & 0.1 & 0.1 \\
0.4 & 0.2 & 0.1 \\
0 & 0.4 & 0.3
\end{bmatrix}
\]

\[
D = \begin{bmatrix}
20 \\
10 \\
30
\end{bmatrix}
\]

(b) (2 points) Which industries in this economy are profitable and why?

All industries are profitable (1pt reason) since each column sums to less than 1.

(c) (2 points) Is this economy productive? Explain your reasoning.

Yes, it is productive since every industry is profitable (1pt reason).

(d) (8 points) How much must each sector produce in order to meet the demand?

\[
(I - C)x = D
\]

\[
(I - C) = \begin{bmatrix}
0.8 & 0 & -0.1 \\
-0.1 & 0.8 & -0.1 \\
-0.1 & -0.4 & 0.7
\end{bmatrix}
\]

\[
x = \begin{bmatrix}
0.8 & 0 & -0.1 \\
-0.1 & 0.8 & -0.1 \\
-0.1 & -0.4 & 0.7
\end{bmatrix}^{-1} \begin{bmatrix}
20 \\
10 \\
30
\end{bmatrix}
\]

\[
= \begin{bmatrix}
3.7 \\
2 \\
1
\end{bmatrix}
\]

\[
(I - C)x = \begin{bmatrix}
0.8 & 0 & -0.1 \\
-0.1 & 0.8 & -0.1 \\
-0.1 & -0.4 & 0.7
\end{bmatrix}^{-1} \begin{bmatrix}
20 \\
10 \\
30
\end{bmatrix}
\]

\[
= \begin{bmatrix}
3.7 \\
2 \\
1
\end{bmatrix}
\]

(e) (3 points) A certain economy is known NOT to be productive. Its consumption matrix \(C^*\) is such that

\[
(I - C^*)^{-1} = \begin{bmatrix}
0.3 & 0.7 & 0.6 \\
0.2 & -0.3 & -0.7 \\
0.2 & -0.3 & -0.4
\end{bmatrix}
\]

Find one societal demand vector that the economy can satisfy.

\[
x = \begin{bmatrix}
1.2 & 0.6 \\
-0.3 & -0.7 \\
-0.2 & -0.4
\end{bmatrix} \begin{bmatrix}
10 \\
10
\end{bmatrix}
\]

\[
= \begin{bmatrix}
16 \\
-3 \\
-2
\end{bmatrix}
\]