This worksheet establishes a number of important properties of prime numbers. After you’ve finished, reflect on how everything depended on the Euclidean algorithm (hence on the division algorithm).

(1) Let $c \in \mathbb{Z}$ be nonzero. Prove that if $c|(ab)$ and $\gcd(a, c) = 1$, then $c|b$.

(2) Let $p$ be a prime number such that $p|(ab)$. Prove that $p|a$ or $p|b$.\(^1\)

(3) Rephrase (2) using congruences.

\(^1\)This is the most important property of a prime number. Note that this doesn’t work for composite numbers $n$, since then we can factor $n = ab$ into smaller pieces $a$ and $b$ that are not divisible by $n$. 
(4) Let \( p \) be a prime number and let \( a \in \mathbb{Z} \) be such that \( a \not\equiv 0 \pmod{p} \). Prove that there is \( x \in \mathbb{Z} \) such that \( ax \equiv 1 \pmod{p} \).

(5) Give an example of \( a, n \in \mathbb{Z} \), with \( n > 0 \), such that \( a \not\equiv 0 \pmod{n} \), but there is no \( x \in \mathbb{Z} \) such that \( ax \equiv 1 \pmod{n} \).

(6) Use (4) to give another proof of (3).