(1) (a) Prove that for any sets $A$, $B$, and $C$, we have

$$(A \cup B) \setminus C \subseteq A \cup (B \setminus C).$$

**Solution.** Let $x \in (A \cup B) \setminus C$. Then $x \in (A \cup B)$ and $x \notin C$. If $x \in A$, then $x \in A \cup (B \setminus C)$. If $x \notin A$, then $x \in B$ since $x \in A \cup B$. Since $x \in B$ and $x \notin C$, $x \in B \setminus C$. Then $x \in A \cup (B \setminus C)$ in this case as well.

Since $x$ was arbitrary, this shows that $(A \cup B) \setminus C \subseteq A \cup (B \setminus C)$. □

(b) Does one always have equality in part (a)? Prove or give a counterexample.

**Solution.** No. Consider the example of $A = B = C = \mathbb{N}$ (any nonempty set would do here). Then $A \cup B = \mathbb{N} \cup \mathbb{N} = \mathbb{N}$, so $(A \cup B) \setminus C = \mathbb{N} \setminus \mathbb{N} = \emptyset$.

On the other hand, $B \setminus C = \mathbb{N} \setminus \mathbb{N} = \emptyset$. So $A \cup (B \setminus C) = \mathbb{N} \cup \emptyset = \mathbb{N}$. □

(2) Let $f : [-1, \infty) \to \mathbb{R}$ be given by $f(x) = x + 1$ and let $g : \mathbb{R} \to [0, \infty)$ be given by $g(x) = 2|x|$ (here $|x|$ is the absolute value of $x$).

(a) Prove that $g \circ f$ is bijective.

**Solution.** First note that the composite is the function $g \circ f : [0, \infty) \to [0, \infty)$ given by the formula $g \circ f(x) = g(f(x)) = g(x + 1) = 2|x + 1|$.

Since $x + 1 \geq 0$ for all $x \in [-1, \infty)$, we see that

$$g \circ f(x) = 2(x + 1)$$

for all $x \in [-1, \infty)$.

We first show that $g \circ f$ is injective. Take $x_1, x_2 \in [-1, \infty)$ such that $g \circ f(x_1) = g \circ f(x_2)$. Then

$$2(x_1 + 1) = 2(x_2 + 1) \implies x_1 + 1 = x_2 + 1 \implies x_1 = x_2.$$ 

This shows that $g \circ f$ is injective.

Now we show that $g \circ f$ is surjective. Take $y \in [0, \infty)$. Since $y \geq 0$, we have $\frac{y}{2} \geq 0$ and $\frac{y}{2} - 1 \geq -1$. Then setting $x = \frac{y}{2} - 1$, we have $x \in [-1, \infty)$ and

$$g \circ f(x) = 2\left(\frac{y}{2} - 1 + 1\right) = y.$$ 

This shows that $g \circ f$ is surjective.

We conclude that $g \circ f$ is bijective. □

(b) Are either of $f$ or $g$ bijective? Explain your answer.

**Solution.** Neither are bijective.

First, since $g(-1) = 2|-1| = 2 = g(1)$, we see that $g$ is not injective.

Secondly, for any $x \geq -1$, $f(x) = x + 1 \geq 0$. So there is no $x \in [-1, \infty)$ such that $f(x) = -1$, and $f$ is not surjective.
(3) Prove that any \( n \in \mathbb{N} \) can be written as a sum of distinct powers of 2, i.e. for all \( n \in \mathbb{N} \), are integers \( e_1 > e_2 > \cdots > e_k \geq 0 \) such that \( n = 2^{e_1} + 2^{e_2} + \cdots + 2^{e_k} \)

(Hint: Note that in this expression, \( e_1 \) is uniquely determined by the fact that \( 2^{e_1} \) is the largest power of 2 less than or equal to \( n \). How might this observation help you in proving the inductive step? What kind of induction should you use?)

**Solution** We prove this using strong induction.

**Base case:** We have \( 1 = 2^0 \), so the result holds for \( n = 1 \).

**Inductive step:** Fix \( n \geq 1 \), and assume the result holds for any \( 1 \leq m \leq n \). We want to prove the result for \( n + 1 \).

Let \( e_1 \in \mathbb{Z}_{\geq 0} \) be the largest nonnegative integer such that \( 2^{e_1} \leq n + 1 \). Thus, \( 2^{e_1} \leq n + 1 \) and \( 2^{e_1+1} > n + 1 \). If \( 2^{e_1} = n + 1 \), then we have written \( n + 1 \) as a sum of distinct powers of 2 (\( k = 1 \) in this case), and the inductive step holds in this case.

Now assume that \( 2^{e_1} < n + 1 \), and let \( m = n + 1 - 2^{e_1} \). Since \( 2^{e_1} < n + 1 \), we have \( m > 0 \), and since \( 2 \cdot 2^{e_1} = 2^{e_1+1} > n + 1 \), we have \( m < 2^{e_1} \). Then \( 1 \leq m \leq n \), and we can apply our inductive hypothesis to \( m \). Thus, there are integers \( e_2 > \cdots > e_k \geq 0 \) such that \( m = 2^{e_2} + \cdots + 2^{e_k} \).

We then have
\[
n + 1 = 2^{e_1} + m = 2^{e_1} + 2^{e_2} + \cdots + 2^{e_k}.
\]

Also, since \( m < 2^{e_1} \), we have \( e_1 > e_2 \). This finishes the proof of the inductive step.

By the principal of strong induction, the theorem is proved. \( \square \)

(4) Let \( U \) be a set and let \( \{A_i : i \in I\} \) be an indexed collection of sets such that \( A_i \subseteq U \) for all \( i \in I \).

(a) Explain why \( U \setminus \bigcup_{i \in I} A_i = \bigcap_{i \in I} (U \setminus A_i) \) and \( U \setminus \bigcap_{i \in I} A_i = \bigcup_{i \in I} (U \setminus A_i) \).

**Solution.** Firstly, \( U \setminus \bigcup_{i \in I} A_i \) is the set of all \( x \in U \) such that \( x \notin \bigcup_{i \in I} A_i \). Now \( x \in U \setminus \bigcup_{i \in I} A_i \) if and only if there exists \( i \in I \) such that \( x \notin A_i \). So \( x \notin \bigcup_{i \in I} A_i \) if and only if \( x \notin A_i \) for all \( i \in I \). Thus the set of \( x \in U \) such that \( x \notin \bigcup_{i \in I} A_i \) equals the set of \( x \in U \) such that \( x \in U \setminus A_i \) for all \( i \in I \), i.e. \( U \setminus (\bigcup_{i \in I} A_i) = \bigcap_{i \in I} (U \setminus A_i) \).

Secondly, \( U \setminus \bigcap_{i \in I} A_i \) is the set of all \( x \in U \) such that \( x \notin \bigcap_{i \in I} A_i \). We have that \( x \in \bigcap_{i \in I} A_i \) if and only if \( x \in A_i \) for all \( i \in I \). So \( x \notin \bigcap_{i \in I} A_i \) if and only if there exists \( i \in I \) such that \( x \notin A_i \). Thus the set of \( x \in U \) such that \( x \notin \bigcap_{i \in I} A_i \) equals the set of \( x \in U \) such that \( x \in U \setminus A_i \) for some \( i \in I \), i.e. \( U \setminus (\bigcap_{i \in I} A_i) = \bigcup_{i \in I} (U \setminus A_i) \).

(b) Consider the example \( U = \mathbb{R} \), \( I = \mathbb{N} \), and for every \( n \in \mathbb{N} \), \( A_n = \{x \in \mathbb{R} : |x| \geq n\} \).

What is \( \bigcup_{n \in \mathbb{N}} A_n \)? What is \( \mathbb{R} \setminus (\bigcup_{n \in \mathbb{N}} A_n) \)?

**Solution.** Note that \( A_n \subseteq A_1 \) for every \( n \in \mathbb{N} \). Thus
\[
x \in \bigcup_{n \in \mathbb{N}} A_n \iff x \in A_n \text{ for some } n \in \mathbb{N} \iff x \in A_1.
\]

So \( \bigcup_{n \in \mathbb{N}} A_n = A_1 = \{x \in \mathbb{R} : |x| \geq 1\} \), and
\[
\mathbb{R} \setminus \bigcup_{n \in \mathbb{N}} A_n = \{x \in \mathbb{R} : |x| < 1\} = (-1, 1).
\]

What is \( \mathbb{R} \setminus A_n \)? What is \( \bigcap_{n \in \mathbb{N}} (\mathbb{R} \setminus A_n) \)? (You should get the same answer as in part (b)).

**Solution.** \( \mathbb{R} \setminus A_n = \{x \in \mathbb{R} : |x| < n\} = (-n, n) \). Note that \( (-1, 1) \subseteq (-n, n) \) for all \( n \in \mathbb{N} \). So
\[
x \in \bigcap_{n \in \mathbb{N}} (-n, n) \iff x \in (-n, n) \text{ for all } n \in \mathbb{N} \iff x \in (-1, 1).
\]

So \( \bigcap_{n \in \mathbb{N}} (\mathbb{R} \setminus A_n) = (-1, 1) \). \( \square \)