This worksheet will be practice on some of the more recent set theoretic construction. Parts of it will also practice dealing with functions, and even one instance of induction.
Heads up: some of these questions are confusing! Ask me (Prof Allen) questions!

1. Cartesian products

(1) Let $A_1$ and $A_2$ be sets. Recall what it means for two ordered pairs $(a_1, a_2)$ and $(b_1, b_2)$ in $A_1 \times A_2$ to be equal.

Solution. $(a_1, a_2) = (b_1, b_2)$ means $a_1 = b_1$ and $a_2 = b_2$.

(2) Let $A_1$, $A_2$, and $A_3$ be sets and consider the iterated Cartesian product $(A_1 \times A_2) \times A_3$. What does it mean for two elements $((a_1, a_2), a_3)$ and $((b_1, b_2), b_3)$ of $(A_1 \times A_2) \times A_3$ to be equal?

Solution.

$$((a_1, a_2), a_3) = ((b_1, b_2), b_3) \iff (a_1, a_2) = (b_1, b_2) \text{ and } a_3 = b_3$$

(3) Let $n \geq 2$ be a natural number and let $A_1, \ldots, A_n$ be sets. Define a function $f : A_1 \times \cdots \times A_n \to (A_1 \times \cdots \times A_{n-1}) \times A_n$ by $f(a_1, \ldots, a_n) = ((a_1, \ldots, a_{n-1}), a_n)$. Prove that $f$ is a bijection.

Solution. We first check injectivity. Let $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in A_1 \times \cdots \times A_n$ be such that $f(a_1, \ldots, a_n) = f(b_1, \ldots, b_n)$. We have

$$f(a_1, \ldots, a_n) = f(b_1, \ldots, b_n) \implies ((a_1, \ldots, a_{n-1}), a_n) = ((b_1, \ldots, b_{n-1}), b_n)$$

$$\implies (a_1, \ldots, a_{n-1}) = (b_1, \ldots, b_{n-1}) \text{ and } a_n = b_n$$

$$\implies a_1 = b_1 \text{ for each } 1 \leq i \leq n$$

$$\implies (a_1, \ldots, a_n) = (b_1, \ldots, b_n).$$

This shows that $f$ is injective.

Now we check surjectivity. Take arbitrary $((a_1, \ldots, a_{n-1}), a_n) \in (A_1 \times \cdots \times A_{n-1}) \times A_n$. Then $(a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n$ satisfies $f(a_1, \ldots, a_n) = ((a_1, \ldots, a_{n-1}), a_n)$. This shows that $f$ is surjective.

Since $f$ is both injective and surjective, it is bijective. □
(4) In class we proved that if \( A_1 \) and \( A_2 \) are finite sets, then \( A_1 \times A_2 \) is finite and \( |A_1 \times A_2| = |A_1||A_2| \). I then asserted that the similar result for any finite number of sets is true.

Prove this formally using induction. That is, prove that for any \( n \in \mathbb{N} \), if \( A_1, \ldots, A_n \) are finite sets, then \( A_1 \times \cdots \times A_n \) is finite and \( |A_1 \times \cdots \times A_n| = |A_1| \cdots |A_n| \).

**Solution.** We prove this by induction on \( n \in \mathbb{N} \). When \( n = 1 \), the statement becomes \( |A_1| = |A_1| \), which is trivially true.

Fix \( n \in \mathbb{N} \), and assume the result is true for \( n \). Let \( A_1, \ldots, A_{n+1} \) be finite sets. By the previous question, there is a bijection
\[
f : A_1 \times \cdots \times A_{n+1} \to (A_1 \times \cdots \times A_n) \times A_{n+1}
\]
so it suffices to prove that
\[
( A_1 \times \cdots \times A_n ) \times A_{n+1}
\]
is finite with cardinality \( |A_1| \cdots |A_{n+1}| \).

By the inductive hypothesis, \( A_1 \times \cdots \times A_n \) is finite with cardinality \( |A_1| \cdots |A_n| \). Note that \( (A_1 \times \cdots \times A_n) \times A_{n+1} \) is the Cartesian product of the two sets \( A_1 \times \cdots \times A_n \) and \( A_{n+1} \). So by what we proved in class and the inductive hypothesis, \( (A_1 \times \cdots \times A_n) \times A_{n+1} \) is finite and has cardinality
\[
|A_1 \times \cdots \times A_n| \cdot |A_{n+1}| = (|A_1| \cdots |A_n|) \cdot |A_{n+1}| = |A_1| \cdots |A_{n+1}|
\]
This finishes the proof of the inductive step.

The result now follows by induction. \( \square \)

2. **Power sets**

(5) Let \( A = \{7, \pi, \{3, 7\}, \mathbb{Q}\} \).

(a) What is \( |\mathcal{P}(A)| \)?

**Solution.** Since \( |A| = 4 \), \( |\mathcal{P}(A)| = 2^4 = 16 \).

(b) Let the elements of \( A \) be ordered as listed above and let \( f : \mathcal{P}(A) \to \{0, 1\}^{|A|} \) be the coding function used in the proof at the beginning of class on Wednesday. (If you don’t remember and don’t have your notes, ask what it is!)

(i) What is \( f(\{\pi, \{3, 7\}\}) \)?

**Solution.** \( f(\{\pi, \{3, 7\}\}) = (0, 1, 1, 0) \).

(ii) What subset \( B \subseteq A \) satisfies \( f(B) = (1, 0, 1, 1) \)?

**Solution.** \( B = \{7, \{3, 7\}, \mathbb{Q}\} \).

(iii) If \( B \subseteq A \) has \( |B| = 2 \), what does that imply about \( f(B) \)?

**Solution.** It implies that \( f(B) \) contains two 0s and two 1s.

(6) Let \( A \) be a set and let \( f : A \to \mathcal{P}(A) \) be a function.

(a) Let \( a \in A \). Does it make sense to say “\( a \in f(a) \)” and “\( a \notin f(a) \)”?

**Solution.** Yes, it does. Since \( f \) is a function with domain \( A \) and codomain \( \mathcal{P}(A) \), for every \( a \in A \), \( f(a) \) is an element of \( \mathcal{P}(A) \), hence a subset of \( A \). It therefore makes sense to write \( a \in f(a) \) and \( a \notin f(a) \).
(b) Consider the following peculiar set: \( B = \{ a \in A : a \notin f(a) \} \); it is a subset of \( A \), and it depends on \( f \).

(i) Say for example, \( f : A \to \mathcal{P}(A) \) is the function \( f(a) = \{ a \} \) for all \( a \in A \). What is \( B \)?

**Solution.** \( B = \{ a \in A : a \notin \{ a \} \} = \emptyset \), since every \( a \in A \) satisfies \( a \in \{ a \} \).

(ii) Let \( C \subseteq A \) be some fixed subset and say for example, \( f : A \to \mathcal{P}(A) \) is the function \( f(a) = C \) for all \( a \in A \). What is \( B \)?

**Solution.** \( B = \{ a \in A : a \notin C \} = A \setminus C \).

(c) Use the peculiar subset in (b) to prove that \( f : A \to \mathcal{P}(A) \) cannot be surjective.

**Solution.** If we show that the set \( B = \{ a \in A : a \notin f(a) \} \) is not in the image of \( f \), then \( f \) is not surjective. Assume, for a contradiction, that \( B \) is in the image of \( f \). Thus, we are assuming there is \( a \in A \) such that \( f(a) = B \).

We consider two cases: \( a \in B \) and \( a \notin B \).

If \( a \in B \), then \( a \in f(a) \) since \( f(a) = B \). By the definition of \( B \), this implies \( a \notin B \), and we have a contradiction.

If \( a \notin B \), then \( a \notin f(a) \) since \( f(a) = B \). By the definition of \( B \), this implies that \( a \in B \), and we again have a contradiction.

Thus, \( B \) is not in the image of \( f \), and \( f \) is not surjective.

**Remark.** The argument used above is the mathematical version of what is known as the Barber’s paradox:

There is a town with one barber. Anyone who does not cut their own hair has their hair cut by the barber, and the barber does not cut the hair of any person who cuts their own hair. Does the barber cut their own hair?

3. Indexed collections of sets

(7) For each \( r \in \mathbb{R} \), let \( A_r = \{(x, y) \in \mathbb{R}^2 : y = rx \} \). Find \( \bigcup_{r \in \mathbb{R}} A_r \) and \( \bigcap_{r \in \mathbb{R}} A_r \). Is this collection pairwise disjoint? Justify your answers (or at least, make sure you know how to).

**Solution.** We first claim that \( \bigcup_{r \in \mathbb{R}} A_r = \{(x, y) \in \mathbb{R}^2 : x \neq 0 \} \cup \{(0, 0)\} \).

First take \((a, b) \in \bigcup_{r \in \mathbb{R}} A_r \). Then there is \( r \in \mathbb{R} \) such that \((a, b) \in A_r \), which means that \( b = ra \). From \( b = ra \), we see that if \( a = 0 \), then \( b = 0 \) as well. In other words, \( a \neq 0 \) or \((a, b) = (0, 0)\). This means \((a, b) \in \{(x, y) \in \mathbb{R}^2 : y \neq 0 \} \cup \{(0, 0)\} \).

Now take \((a, b) \in \{(x, y) \in \mathbb{R}^2 : x \neq 0 \} \cup \{(0, 0)\} \). If \( a = 0 \), then \( b = 0 \) as well, and \((a, b) = (0, 0) \in A_r \) for every \( r \in \mathbb{R} \). If \( a \neq 0 \), then \((a, b) \in A_r \) for \( r = \frac{b}{a} \) since \( b = \left( \frac{b}{a} \right) a \). So in either case, \((a, b) \in \bigcup_{r \in \mathbb{R}} A_r \).

Since we have shown both containments, we conclude that

\[
\bigcup_{r \in \mathbb{R}} A_r = \{(x, y) \in \mathbb{R}^2 : x \neq 0 \} \cup \{(0, 0)\}.
\]

Next we claim that \( \bigcap_{r \in \mathbb{R}} A_r = \{(0, 0)\} \). For every \( r \in \mathbb{R} \), we have \( 0 = r \cdot 0 \), so \((0, 0) \in A_r \) for every \( r \in \mathbb{R} \). This shows that \( \{(0, 0)\} \subseteq \bigcap_{r \in \mathbb{R}} A_r \).

Now take \((a, b) \in \bigcap_{r \in \mathbb{R}} A_r \), so \((a, b) \in A_r \) for every \( r \in \mathbb{R} \). This means that \( b = ra \) for every \( r \in \mathbb{R} \). Taking \( r = 0 \), we see that \( b = 0 \). Taking \( r = 1 \), we see that \( b = a \). Together, these imply that \((a, b) = (0, 0)\). This shows that \( \bigcap_{r \in \mathbb{R}} A_r \subseteq \{(0, 0)\} \).

Since we have shown both containments, we conclude that

\[
\bigcap_{r \in \mathbb{R}} A_r = \{(0, 0)\}.
\]

This collection is not pairwise disjoint since any two sets in this collection share the common element \( \{(0, 0)\} \).
(8) For each $n \in \mathbb{Z}_{\geq 0}$, let $A_n = \{x \in \mathbb{R} : x \geq n\}$.

(a) Explain why $\bigcap_{n=0}^{\infty} A_n = \emptyset$.

Solution. Since $A_n \subseteq \mathbb{R}$ for all $n \in \mathbb{Z}_{\geq 0}$, we see that $\bigcap_{n=0}^{\infty} A_n \subseteq \mathbb{R}$. If $x \in \bigcap_{n=0}^{\infty} A_n$, then $x \in A_n$ for every $n \in \mathbb{Z}_{\geq 0}$, i.e. $x \geq n$ for every $n \geq 0$. But for any real number $x$, we can find some non-negative integer $n$ such that $n > x$, so we must have $\bigcap_{n=0}^{\infty} A_n = \emptyset$. □

(b) Show that for any finite subset $I \subseteq \mathbb{Z}_{\geq 0}$, $\bigcap_{i \in I} A_i \neq \emptyset$.

Solution. Let $I \subseteq \mathbb{Z}_{\geq 0}$ be a finite set. Since $I$ is finite, it has a maximum element; let $n$ be this maximum element of $I$. So $i \leq n$ for every $i \in I$, and any real number $x$ with $x \geq n$, satisfies $x \geq i$ for all $i \in I$. So for any real number $x$ with $x \geq n$, we have $x \in \bigcap_{i \in I} A_i$. □

(9) Let $A$ be a set.

(a) Consider the indexed collection $\{B : B \in \mathcal{P}(A)\}$. What is $\bigcup_{B \in \mathcal{P}(A)} B$?

Solution. We claim that $\bigcup_{B \in \mathcal{P}(A)} B = A$.

To see this, note first that if $x \in \bigcup_{B \in \mathcal{P}(A)} B$, then there is $B \in \mathcal{P}(A)$ such that $x \in B$. Since $B \subseteq A$, we have $x \in A$. This shows that $\bigcup_{B \in \mathcal{P}(A)} B \subseteq A$.

Now take $a \in A$. To show that $a \in \bigcup_{B \in \mathcal{P}(A)} B$, we need to show that there is $B \in \mathcal{P}(A)$ such that $a \in B$. This is true, since $A \in \mathcal{P}(A)$ and $a \in A$. This shows that $A \subseteq \bigcup_{B \in \mathcal{P}(A)} B$.

We conclude that $\bigcup_{B \in \mathcal{P}(A)} B = A$. □

(b) Consider the indexed collection $\{\{B\} : B \in \mathcal{P}(A)\}$. What is $\bigcup_{B \in \mathcal{P}(A)} \{B\}$?

Solution. We claim that $\bigcup_{B \in \mathcal{P}(A)} \{B\} = \mathcal{P}(A)$.

To see this, note first that if $x \in \bigcup_{B \in \mathcal{P}(A)} \{B\}$, there is $B \in \mathcal{P}(A)$ such that $x \in B$. But since $B$ is the only element of the set $\{B\}$, this implies that $x = B$. So $x \in \mathcal{P}(A)$. This shows that $\bigcup_{B \in \mathcal{P}(A)} \{B\} \subseteq \mathcal{P}(A)$.

Now take $C \in \mathcal{P}(A)$. To show that $C \in \bigcup_{B \in \mathcal{P}(A)} \{B\}$, we want to show that there is $B \in \mathcal{P}(A)$ such that $C \in \{B\}$. Taking $B = C$, we see that this is true. This shows that $\mathcal{P}(A) \subseteq \bigcup_{B \in \mathcal{P}(A)} \{B\}$.

We conclude that $\bigcup_{B \in \mathcal{P}(A)} B = \mathcal{P}(A)$. □

Remark. More generally, if $I$ is any set, then $\bigcup_{i \in I} \{i\} = I$. The proof is the same as above.