The worksheet proves the convergent halves of the comparison and ratio tests. Before starting, you should review the definitions of Cauchy sequences and convergent series, and recall that a sequence of real numbers converges if and only if it is Cauchy.

**Theorem** (Comparison test). Let \((x_k)_{k=1}^\infty\) and \((y_k)_{k=1}^\infty\) be sequences such that \(y_k \geq 0\) for all \(k \in \mathbb{N}\).
1. If \(|x_k| \leq y_k\) for all \(k \in \mathbb{N}\) and \(\sum_{k=1}^\infty y_k\) converges, then \(\sum_{k=1}^\infty x_k\) converges.
2. If \(y_k \leq x_k\) for all \(k \in \mathbb{N}\) and \(\sum_{k=1}^\infty y_k\) diverges, then \(\sum_{k=1}^\infty x_k\) diverges.

Note that by taking \(y_k = |x_k|\), we obtain the following useful corollary:

**Corollary 1.** If \(\sum_{k=1}^\infty |x_k|\) converges, then so does \(\sum_{k=1}^\infty x_k\).

1. Prove part 1 of the comparison test. (Hint: Think in terms of the partial sums being Cauchy.)

2. Fix \(b \in \mathbb{N}\) with \(b \geq 2\). For each \(k \in \mathbb{N}\), let \(d_k \in \{0, 1, \ldots, b - 1\}\). Prove that \(\sum_{k=1}^\infty \frac{d_k}{b^k}\) converges to some real number in \([0, 1]\).
Theorem (The ratio test). Let \((x_k)_{k=1}^{\infty}\) be a sequence with \(x_k \neq 0\) for all \(k \in \mathbb{N}\) and such that \(|\frac{x_{k+1}}{x_k}|\) converges. Let \(r = \lim_{k \to \infty} |\frac{x_{k+1}}{x_k}|\).

1. If \(r < 1\), then \(\sum_{k=1}^{\infty} x_k\) converges.
2. If \(r > 1\), then \(\sum_{k=1}^{\infty} x_k\) diverges.

(3) Prove part 1 of the ratio test by following the steps below.

(a) Explain why we can assume that \(x_k > 0\) for all \(k \in \mathbb{N}\). (Hint: Turn the page over.)

Fix \(\epsilon > 0\). For \(n \in \mathbb{N}\), let \(s_n\) be the \(n\)th partial sum. Fix also a real number \(t\) with \(r < t < 1\).

(b) Explain why there is \(K_1 \in \mathbb{N}\) such that for all \(k \geq K_1\), we have \(\frac{x_{k+1}}{x_k} < t\). Deduce that for any \(k \geq K_1\) and \(j \geq 1\), we have \(x_{k+j} < x_k t^j\). (Strictly speaking this should be proved by induction on \(j\), but you can ignore that for now.)

(c) Show that \(\lim_{k \to \infty} x_k = 0\). Deduce that we can find \(K_2 \in \mathbb{N}\) such that for all \(k \geq K_2\), we have \(x_k < \epsilon(1 - t)\).

(d) Let \(N = \max\{K_1, K_2\}\). Show that for any \(n \geq m \geq N\), we have \(0 < s_n - s_m < \epsilon(1 - t) \sum_{j=1}^{\infty} t^j\). What does this last expression equal?

(e) Conclude.