Theorem (Completeness of IR) A sequence \((x_n)_{n=1}^{\infty}\) of real numbers converges \(\iff\) it is Cauchy.

Proof. First assume \((x_n)_{n=1}^{\infty}\) is convergent and let \(L = \lim_{n \to \infty} x_n\).

Fix \(\varepsilon > 0\). There exists \(N \in \mathbb{N}\) such that for all \(n \geq N\), we have
\[ |x_n - L| < \frac{\varepsilon}{2}, \]

Then, \(A n, m \geq N,\) we have
\[ |x_n - x_m| = |x_n - L + L - x_m| \leq |x_n - L| + |L - x_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \]

This shows that \((x_n)_{n=1}^{\infty}\) is Cauchy.

Now assume that \((x_n)_{n=1}^{\infty}\) is Cauchy. By the lemma, it is bounded. By the Bolzano-Weierstrass theorem, we can find a convergent subsequence \((x_{k_n})_{n=1}^{\infty}\) of \((x_n)_{n=1}^{\infty}\).

Let \(L = \lim_{n \to \infty} x_{k_n}\). We'll show that \(\lim_{n \to \infty} x_n = L\).

Fix \(\varepsilon > 0\). Since \((x_n)_{n=1}^{\infty}\) is Cauchy, there exists \(N_0 \in \mathbb{N}\) such that for all \(n, m \geq N_0\), we have
\[ |x_n - x_m| < \frac{\varepsilon}{2}, \]

there exists \(N \in \mathbb{N}\) such that for all \(n \geq N\), we have
\[ |x_n - L| < \frac{\varepsilon}{2}. \]

Since \(x_{k_n} = x_{n_k}\) for some increasing sequence of indices \(n_1 < n_2 < \ldots < n_k \ldots\), we can choose some \(k_0 \geq k\) such that \(n_k \geq N_0\).

Letting \(N = n_{k_0}\), we have that for all \(n \geq N_0\),
\[ |x_n - L| = |x_n - x_{n_k} + x_{n_k} - L| \leq |x_n - x_{n_k}| + |x_{n_k} - L| \leq |x_n - x_{n_k}| + \frac{\varepsilon}{2} \leq \varepsilon. \]

So \((x_n)_{n=1}^{\infty}\) is convergent.

\[ \square \]

Remark. Note that convergent \(\iff\) Cauchy uses only the definitions, but Cauchy \(\iff\) convergent uses the lemma.
the Bolzano-Weierstrass Theorem, which uses the monotone convergence theorem, which uses the least upper bound property, which we have been taking as an axiom. One can turn this around, taking the completeness of \( \mathbb{R} \) as an axiom, using it to prove the Bolzano-Weierstrass Theorem, using Bolzano-Weierstrass to prove the monotone convergence theorem, and using the monotone convergence theorem to prove the least upper bound property. So in some sense, the least upper bound property and the completeness of \( \mathbb{R} \) are equivalent.

Aside: We noticed previously that \( \mathbb{R} \) can actually be constructed as Dedekind cuts, and that using this construction, the least upper bound property can be proved (i.e. it is not an axiom). There is an alternate construction of \( \mathbb{R} \) as follows. Let \( A \) be the set of all Cauchy sequences of rational numbers. Define an equivalence relation \( \sim \) on \( A \) by \( (a_n) \sim (b_n) \iff \lim_{n \to \infty} |a_n - b_n| = 0 \). Then we can define \( \mathbb{R} = A / \sim \). With this definition, it is easier to first prove the completeness of \( \mathbb{R} \) and then use it to prove the least upper bound property.

We can apply what we've been doing to define infinite series.

**Definition:** Let \( (x_n)_{n=1}^{\infty} \) be a sequence of real numbers. The formal expression \( \sum_{n=1}^{\infty} x_n \) is called an infinite series. For each \( n \in \mathbb{N} \), \( s_n = \sum_{k=1}^{n} x_k \) is the \( n \)-th partial sum. If \( (s_n)_{n=1}^{\infty} \) converges, we say that \( \sum_{n=1}^{\infty} x_n \) converges, otherwise we say it diverges. If \( L = \lim_{n \to \infty} s_n \), we write \( \sum_{n=1}^{\infty} x_n = L \).

Now to examples.
Example (The geometric series) For any $x \in \mathbb{R}$, $\sum_{k=1}^{\infty} x^k = \frac{x}{1-x}$ if $|x| < 1$ and diverges otherwise.

**Proof** For each $n \in \mathbb{N}$, let $s_n = \sum_{k=1}^{n} x^k$.

**Case 1:** $x = 1$. Then $s_n = n$ and $(s_n)_{n=1}^{\infty}$ does not converge.

**Case 2:** $x = -1$. Then $s_n = \sum_{k=1}^{n} (-1)^{k+1}$ if $n$ is odd, and $(s_n)_{n=1}^{\infty}$ does not converge.

**Case 3:** $|x| < 1$. We proceed $\lim_{n \to \infty}$ (when doing induction) that

$$\sum_{k=0}^{n} x^k = \frac{x^{n+1}-1}{x-1}, \quad s_n = \sum_{k=1}^{n} x^k = \frac{x^{n+1}-1}{x-1} - \frac{1-x}{x-1}$$

First we claim that $\lim_{x \to 0} x^k = 0$. Fix $0 < \epsilon < 1$. Then

$$\log\left(\frac{1}{\epsilon}\right) > 0$$

So $\exists K \in \mathbb{N}$ such that $K \log\left(\frac{1}{x}\right) > \log\left(\frac{1}{\epsilon}\right)$ by the Archimedean principle. Then $\forall k \geq K$, we have $k \log\left(\frac{1}{x}\right) > \log\left(\frac{1}{\epsilon}\right) \Rightarrow \left(\frac{1}{x}\right)^k > \frac{1}{\epsilon} = |x|^k < \epsilon$.

So $\lim_{n \to \infty} x^n = 0$. Then by limit laws

$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{x^{n+1}-1}{x-1} = \frac{0}{x-1} = \frac{x}{1-x}$$

**Case 3:** $|x| > 1$. We just sketch the proof of this case. An argument using the Archimedean principle and logarithms (similar to above) shows that $(s_n)_{n=1}^{\infty}$ is unbounded. Then a little algebra shows that

$$(s_n)_{n=1}^{\infty} = \left(\frac{x^{n+1}-x}{x-1}\right)_{n=1}^{\infty}$$

is unbounded, so cannot converge. $\Box$
Eq \( \sum_{k=1}^{\infty} \frac{1}{k^2} \) is convergent.

**Proof**

For each \( n \in \mathbb{N} \), let \( S_n = \sum_{k=1}^{n} \frac{1}{k^2} \). Then \( S_{n+1} = S_n + \frac{1}{(n+1)^2} \).

For all \( n \in \mathbb{N} \), we see that \((S_n)_{n=1}^{\infty}\) is nondecreasing.

For all \( n \in \mathbb{N} \),
\[
0 \leq S_n = 1 + \sum_{k=2}^{n} \frac{1}{k^2} < 1 + \sum_{k=2}^{n} \frac{1}{k(k-1)} = 1 + \frac{n-1}{n} \leq 2.
\]

By the monotone convergence theorem, \( \sum_{k=1}^{\infty} \frac{1}{k^2} \) is convergent.

**Remark**

In fact, \( \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \), but this is not obvious!

Note that in the above example, with \( \sum_{k=1}^{\infty} x_k \) convergent,

we have \( \lim_{n \to \infty} x_n = 0 \). This is true more generally:

Instead, let \( \sum_{k=1}^{\infty} x_k \) be a convergent infinite series, and for each \( n \in \mathbb{N} \), let \( S_n = \sum_{k=1}^{n} x_k \) and let \( L = \sum_{k=1}^{\infty} x_k = \lim_{n \to \infty} S_n \).

Then \( L = \lim_{n \to \infty} S_{n+1} \), and by limit laws:
\[
0 = L - L = \lim_{n \to \infty} S_{n+1} - \lim_{n \to \infty} S_n = \lim_{n \to \infty} (S_{n+1} - S_n) = \lim_{n \to \infty} x_{n+1}.
\]

So \( \lim_{n \to \infty} x_n = 0 \) as well.

The converse is not true as the following example shows.

**Example (Harmonic series)** \( \sum_{k=1}^{\infty} \frac{1}{k} \) diverges.

**Proof** Consider the following sequence
\[
(x_k)_{k=1}^{\infty} = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^1}, \frac{1}{2^3}, \frac{1}{2^3}, \ldots, \frac{1}{2^3}, \frac{1}{2^4}, \ldots)
\]

Precisely, for each \( j \in \mathbb{N} \), we list \( x_k = \frac{1}{2^j} \) for \( 2^{j-1} \leq k \leq 2^j - 1 \).
In particular, $x_k \leq \frac{1}{k}$ for any $k \in \mathbb{N}$. For $n \in \mathbb{N}$, let

$$s_n = \sum_{k=1}^{n} x_k.$$

Then,

$$s_1 = \frac{1}{2}, \quad s_2 = \frac{1}{2} + 2 \left(\frac{1}{4}\right) = 2 \left(\frac{1}{2}\right), \quad s_7 = \frac{1}{2} + 2 \left(\frac{1}{4}\right) + 4 \left(\frac{1}{8}\right) = 3 \left(\frac{1}{2}\right)$$

Can show by induction that

$$s_1 \leq \frac{1}{2}$$

Then for $j \in \mathbb{N}$,

$$\frac{j}{2} = \frac{s_j}{s_1} \leq \frac{2^n - 1}{2^n} + \frac{1}{2^n}$$

So the partial sums $s_n$ are unbounded. It follows that $\sum_{k=1}^{\infty} \frac{1}{k}$ is divergent.

\qed