Topics:
- Bounded sequences
- The monotone convergence theorem
- The squeeze theorem

We say a sequence $(x_n)_{n=1}^\infty$ is bounded if $\exists M \in \mathbb{R}$ such that $|x_n| \leq M$ $\forall n \in \mathbb{N}$.

But this is equivalent to the set $\{x_n : n \in \mathbb{N}\}$ being bounded above and below.

- Eq. $(\frac{1}{n})_{n=1}^\infty$ is bounded since $\frac{1}{n} \leq 1$ $\forall n \in \mathbb{N}$
- Eq. $(\frac{n}{n^2})_{n=1}^\infty$ is bounded since $\frac{n}{n^2} = \frac{1}{n} \leq 1$ $\forall n \in \mathbb{N}$
- Eq. $((-1)^n)_{n=1}^\infty$ is bounded since $|(-1)^n| \leq 1$ $\forall n \in \mathbb{N}$

NonEq. $(n)_{n=1}^\infty$ is not bounded by the Archimedean principle.

A convergent sequence is bounded.

Proof let $(x_n)_{n=1}^\infty$ be a convergent sequence and let $L = \lim_{n \to \infty} x_n$.

Then $\exists N \in \mathbb{N}$ such that $\forall n > N$, $|x_n - L| < 1$.

So $\forall n > N$,

$-1 \leq x_n - L \leq 1 \implies -1 + L \leq x_n \leq 1 + L$

$\implies |x_n| \leq |L| + 1$

Hence,

$M = \max\{ |x_1|, |x_2|, \ldots, |x_{N-1}|, (|L|+1)^3 \}$

we have $|x_n| \leq M$ $\forall n \in \mathbb{N}$. □
The converse is not true, as the example $((-1)^n)_{n=1}^\infty$ shows.

**Def:** We say a sequence $(x_n)_{n=1}^\infty$ is
- **monotone increasing** if $x_n \leq x_{n+1}$, $\forall n \in \mathbb{N}$
- **monotone decreasing** if $x_n \geq x_{n+1}$, $\forall n \in \mathbb{N}$
- **monotone** if it is either monotone increasing or monotone decreasing

**Ex:** $(\frac{1}{n})_{n=1}^\infty$ monotone decreasing

$(\frac{1}{n})_{n=1}^\infty$ is not monotone

**The Monotone Convergence Theorem:**
A bounded monotone sequence $(x_n)_{n=1}^\infty$ is convergent.

If $(x_n)_{n=1}^\infty$ is monotone increasing, then $\lim_{n \to \infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$

If $(x_n)_{n=1}^\infty$ is monotone decreasing, then $\lim_{n \to \infty} x_n = \inf\{x_n : n \in \mathbb{N}\}$

**Proof:** Assume $(x_n)_{n=1}^\infty$ is bounded and monotone increasing,

Let $L = \sup\{x_n : n \in \mathbb{N}\}$. Fix $\varepsilon > 0$.

Since $L$ is the least upper bound for $(x_n : n \in \mathbb{N})$, if $N \in \mathbb{N}$ such that $L - \varepsilon < x_N \leq L$

Since $(x_n)_{n=1}^\infty$ is monotone increasing, $\forall n \geq N, x_n \leq L$

$\forall n \geq N, \lim_{n \to \infty} x_n = L$

$L - \varepsilon < x_n \leq L \Rightarrow |x_n - L| < \varepsilon$

**Ex:** Define a sequence $(x_n)_{n=1}^\infty$ by $x_1 = 1$ and $x_{n+1} = 1 + \frac{x_n}{2}$, $\forall n \geq 1$. Let's prove that it is convergent.

*Scratch work:* $x_1 = 1$, $x_2 = 1 + \frac{1}{2} = \frac{3}{2}$, $x_3 = 1 + \frac{3}{4} = \frac{7}{4}$, $x_4 = 1 + \frac{7}{8} = \frac{15}{8}$, $x_5 = 1 + \frac{15}{16} = \frac{31}{16}$
Looks like it is increasing and bounded above by 2.

Maybe can prove by induction:

\[ x_n < 2 \Rightarrow 1 + \frac{x_n}{2} \leq 1 + \frac{2}{2} = 2 \]

Increasing?

\[ x_{n+1} > x_n \Leftrightarrow 1 + \frac{x_n}{2} > x_n \Leftrightarrow 1 + \frac{x_n}{2} > 2 \]

\[ \frac{x_n}{2} < 1 \]

\[ \frac{x_n}{2} \leq 1 \Rightarrow x_n \leq 1 + \frac{x_n}{2} = x_{n+1} \]

Do so: First prove by induction on \( n \geq 1 \) that

\[ x_n \leq 2 \]

When \( n = 1 \), \( x_1 - x_1 = 1 \leq 2 \), so the base case holds.

Take some \( n \geq 1 \) and assume \( x_n \leq 2 \). Then

\[ x_{n+1} = 1 + \frac{x_n}{2} \leq 1 + \frac{2}{2} = 2 \]

which shows the inductive step. By induction, \( x_n \leq 2 \) for all \( n \geq 1 \).

In particular, \( (x_n)_{n=1}^\infty \) is bounded.

Now, \( \forall n \geq 1 \), since \( x_n \leq 2 \), we have

\[ \frac{x_n}{2} \leq 1 \Rightarrow x_n \leq 1 + \frac{x_n}{2} = x_{n+1} \]

So, \( (x_n)_{n=1}^\infty \) is monotonically increasing.

By the monotone convergence theorem, \( (x_n)_{n=1}^\infty \) is convergent.

Another useful tool is

The (The squeeze theorem). Let \((a_n)_{n=1}^\infty\), \((b_n)_{n=1}^\infty\), and \((x_n)_{n=1}^\infty\)
be sequences of real numbers. Assume that \( \forall n \geq 1 \)
\[ a_n \leq x_n \leq b_n \]

and that \((a_n)_{n=1}^\infty\) and \((b_n)_{n=1}^\infty\) are both convergent with
the same limit L. Then \((x_n)_{n=1}^\infty\) is convergent and

\[ \lim_{n \to \infty} x_n = L \]

Proof: Fix \( \varepsilon > 0 \). \( \exists N_1, N_2 \in \mathbb{N} \) such that

\[ \forall n \geq N_1, |a_n - L| < \varepsilon \] \( \quad \) and \( \forall n \geq N_2, |b_n - L| < \varepsilon \).

Let \( N = \max\{N_1, N_2\} \) Then \( \forall n \geq N \)

\[ -\varepsilon \leq a_n - L \leq x_n - L \leq b_n - L \leq \varepsilon \]

\[ \Rightarrow |x_n - L| < \varepsilon \].
Eq. \( \left( \frac{n^4 - n^2 + 1}{n^4 + n^2} \right)^n \) is convergent with limit 1.

Proof. \( \forall n \geq 1 \), we have \(-n^2 + 1 \leq n^2\), hence \( \frac{n^4 - n^2 + 1}{n^4 + n^2} \leq \frac{n^4 - n^2}{n^4 + n^2} \) = \( \frac{n^2 - 1}{n^2 + 1} \).

On the other hand,

\[
\frac{n^2 - 1}{n^2 + 1} = \frac{n^4 - n^2 + 1}{n^4 + n^2} \leq 1
\]

Thus, \( \frac{n^2 - 1}{n^2 + 1} \leq \frac{n^4 - n^2 + 1}{n^4 + n^2} \leq 1 \) \( \forall n \geq 1 \).

We saw previously that

\[
\lim_{n \to \infty} \frac{n^2 - 1}{n^2 + 1} = 1
\]

So by the Squeeze Theorem,

\[
\lim_{n \to \infty} \frac{n^4 - n^2 + 1}{n^4 + n^2} = 1
\]