**Math 347 - Lecture 32**

**Topics - Infima and Supremums**

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**Def** Let \( A \subseteq \mathbb{R} \).

- An upper bound for \( A \) is \( y \in \mathbb{R} \) such that \( \forall x \in A \), \( x \leq y \). If an upper bound exists for \( A \), we say \( A \) is bounded above. An upper bound that is \( \leq \) any other upper bound is called a least upper bound or supremum for \( A \), and is written \( \sup A \).

- A lower bound for \( A \) is \( x \in \mathbb{R} \) such that \( \forall a \in A \), \( x \leq a \). If a lower bound exists for \( A \), we say that \( A \) is bounded below. A lower bound that is \( \geq \) any other lower bound for \( A \) is called a greatest lower bound or infimum for \( A \), and is written \( \inf A \).

**Ex** \((-\infty, 7] \) is bounded above but not bounded below. \( \sup (-\infty, 7] = 7 \)

**Ex** \((3, 7) \) is bounded above and below, \( \sup (3, 7) = 7 \), and \( \inf (3, 7) = 3 \).

Let's prove that \( \sup (3, 7) = 7 \). \( \forall x \in (3, 7), \ x < 7 \), \( \infty \) \( 7 \) is an upper bound. To show it is the least upper bound, it is equivalent to show that \( \forall x < 7 \), \( x \) is not an upper bound for \( (3, 7) \).

Take any \( x < 7 \). Then \( x < \frac{x + 7}{2} < 7 \).

Set \( y = \begin{cases} \frac{x + 7}{2} & \text{if } x > 3 \\ 4 & \text{if } x \leq 3 \end{cases} \).

Then \( y \in (3, 7) \) and \( x < y \), so \( x \) is not an upper bound. \( \square \)
Rule 1. \( \sup A \) and \( \inf A \), if they exist, are unique. Indeed, if \( r \) and \( s \) are two least upper bounds, then \( r \leq s \) and \( s \leq r \), so \( r = s \).

2. As the 2nd example above shows, \( \sup A \) (or \( \inf A \)) does not have to belong to \( A \).

In (Axiom (Least upper bound property)) Any nonempty bounded above subset of \( \mathbb{R} \) has a least upper bound.

\( \sqrt{2} \) exists in \( \mathbb{R} \).

Proof Let \( A = \{ x \in \mathbb{R} : x^2 < 2 \} \) and let \( \alpha = \sup A \).

We'll prove that \( \alpha^2 = 2 \).

Assume otherwise, so either \( \alpha^2 < 2 \) or \( \alpha^2 > 2 \).

Since \( \alpha \) is an upper bound for \( A \), we have \( \alpha > 0 \) (if \( \alpha \in A \), for example). So \( \frac{3}{\alpha} > 0 \) and \( \alpha \neq \frac{3}{\alpha} \).

Consider \( y = \frac{1}{2} \left( \alpha + \frac{3}{\alpha} \right) \).

As theorem from lecture 4 (arithmetic vs geometric mean):

\[
y^2 = \alpha \left( \frac{3}{\alpha} \right) = 2, \quad \text{and} \quad 2 > \frac{3}{\alpha} \cdot \alpha = (\frac{3}{\alpha})^2.
\]

If \( \alpha^2 > 2 \), then \( \alpha > \frac{3}{\alpha} \), which implies that \( y < \alpha \).

But \( y^2 > 2 \) and \( y < \alpha \) contradicts the fact that \( \alpha \) is the least upper bound.

If \( \alpha^2 < 2 \), then \( \alpha < \frac{3}{\alpha} \) and \( y < \frac{3}{\alpha} \).

But the \( \alpha < \frac{3}{\alpha} \) and \( (\frac{3}{\alpha})^2 < 2 \) contradicts the fact that \( \alpha \) is an upper bound for \( A \).

In either case we have a contradiction, and we conclude that \( \alpha^2 = 2 \). \( \square \)
\[
\min \{ \alpha, 3 \} = 3 \quad \max \{ \alpha, 3 \} = \alpha
\]

\[
y = \frac{1}{2} (x + \frac{3}{2})
\]

The \( x^2 < 2 \) \Rightarrow \( 2x^2 < 3 \Rightarrow x < \frac{3}{\sqrt{2}} \) \( x > \frac{3}{\sqrt{2}} \) \( \alpha \) not an upper bound \( \frac{3}{\sqrt{2}} > \alpha \) \( \alpha \) ad in \( A \).

Rul: \( \mathbb{Q} \) does not satisfy the least upper bound property. For example: \( A = \mathbb{Q} - \{ \sqrt{2} \} x^2 < 2 \mathbb{Q} \) is bounded but does not have a least upper bound \( \mathbb{Q} \). Indeed, the argument above shows that if it did, its square would be 2. But we showed in lecture 3 that \( \sqrt{2} \) is no such rational number.

The Archimedean property of \( \mathbb{R} \) \( \mathbb{N} \) is not bounded in \( \mathbb{R} \).

**Proof:** Assume it is. Then by the least upper bound property, \( \alpha = \sup \mathbb{N} \) exists. Since \( \alpha \) is the least upper bound, \( \alpha - 1 \) is not an upper bound for \( \mathbb{N} \). Thus \( \exists n \in \mathbb{N} \) such that \( n > \alpha - 1 \). Thus \( n + 1 \in \mathbb{N} \) and \( n + 1 > \alpha \), a contradiction.

**Cor:** For any positive real numbers \( x, y \) \( \exists n \in \mathbb{N} \) such that \( nx > y \).

**Proof:** Since \( \mathbb{N} \) does not have an upper bound, \( \exists n \in \mathbb{N} \) such that \( n > \frac{y}{x} \).

Taking \( y = 1 \) in this corollary yields the useful special case.
For any positive real number \( x \), \( \exists n \in \mathbb{N} \) such that \( \frac{1}{n} < x \).

Very important: For what we will be doing is the absolute value:
\[ |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \]

For any \( x, y \in \mathbb{R} \), we interpret \( |x - y| \) as the distance between \( x \) and \( y \). In particular, \( |x - y| < \varepsilon \) small \( \Rightarrow \) \( x \) and \( y \) are close.

We will usually use \( \varepsilon \) to denote a positive real number that you should think of as being small.

So for \( \varepsilon > 0 \),
\[ |x - y| < \varepsilon \iff -\varepsilon < x - y < \varepsilon \]
\[ \iff -\varepsilon < y - x < \varepsilon \]
\[ \iff \varepsilon < x - y < \varepsilon \]
\[ \iff x \in (y - \varepsilon, y + \varepsilon) \]

Easy Theorem: Let \( x, y \in \mathbb{R} \). Assume \( x, y \in \mathbb{R}, \varepsilon > 0 \).
We have \( |x - y| < \varepsilon \). Then \( x = y \).

Proof: If \( x \neq y \), then \( |x - y| > 0 \), so taking \( \varepsilon = |x - y| > 0 \),
we have \( |x - y| \neq \varepsilon \).

Yoga of analysis: Often prove \( x = y \) by proving \( |x - y| < \varepsilon \)
for any \( \varepsilon > 0 \). We also use this to motivate definitions:
Say we have STUFF (say a sequence, a function) that
we want to say approaches a real number \( y \).
We make a definition of \( \varepsilon \)-form: \( \forall \varepsilon > 0 \)
we can ensure
\[ |(\text{things from stuff}) - y| < \varepsilon \]
We say a sequence of real numbers \((x_n)_{n=1}^{\infty}\) has a limit \(L \in \mathbb{R}\) if \(\forall \varepsilon > 0, \exists N \in \mathbb{N}\) such that:
\[
  n \geq N \implies |x_n - L| < \varepsilon.
\]
In this case, we write \(\lim_{n \to \infty} x_n = L\) or sometimes just \(x_n \to L\). We say a sequence is convergent if it has a limit.