Recall · If A is a finite set, then its cardinality \(|A|\) is the number of elements it contains.

- If A and B are finite sets, then
  - \(|A| \leq |B| \iff \exists \text{ an injection } f: A \to B\)
  - \(|A| = |B| \iff \exists \text{ a bijection } f: A \to B\)

We use this as the general definition:

**Def:** The cardinalities of sets A and B are written as \(|A|\) and \(|B|\) and they are compared as follows:

- \(|A| \leq |B| \iff \exists \text{ an injection } f: A \to B\)
- \(|A| = |B| \iff \exists \text{ a bijection } f: A \to B\)
- \(|A| < |B| \iff |A| \leq |B| \text{ and } |A| \neq |B|\), i.e. \exists \text{ an injection } f: A \to B \text{ but } \nexists \text{ a bijection } g: A \to B\)

Then on any collection of sets, the relation \(A \sim B \iff |A| = |B|\) is an equivalence relation.

**Rmk** Thus one way to think of \(|A|\) is as the equivalence class of this relation.

**Eg** The function \(f: \mathbb{Z}_0 \to \mathbb{N}\) given by \(f(n) = n + 1\) is a bijection, so \(|\mathbb{Z}_0| = |\mathbb{N}|\) (Hint: \#(\mathbb{Z}_0) = \#(\mathbb{N})

**Eg** Let \(2\mathbb{N}\) be the set of even natural numbers. Then \(f: \mathbb{N} \to 2\mathbb{N}\) given by \(f(n) = 2n\) is a bijection. Thus \(|\mathbb{N}| = |2\mathbb{N}|\)
Notation: The cardinality of \( N \) is denoted \( \aleph_0 \), aleph-null.

**Remark:** In "cardinal arithmetic," the 2 previous examples say that:

\[
\aleph_0 + 1 = \aleph_0 \\
2 \cdot \aleph_0 = \aleph_0
\]

**Def.** We say that a set \( A \) is countable, infinite, or denumerable if \( |A| = \aleph_0 \).

**Remark.** \( |A| = \aleph_0 \iff \exists \text{ a bijection } f: \mathbb{N} \to A \). Setting \( a_n = f(n) \in A \) for each \( n \in \mathbb{N} \), you can think of \( \aleph_0 \)-denumerable as meaning that you can list the elements of \( A \):

\[
A = \{a_1, a_2, a_3, a_4, \ldots \}
\]

in the same way as \( \mathbb{N} \).

Then \( \mathbb{Z} \) is denumerable.

**Idea.** \( \mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \} \)

How can we write these in a list:

\[
\mathbb{Z} = \{a_1, a_2, a_3, \ldots \}
\]

As follows:

\[
\mathbb{Z} = \{0, -1, 1, -2, 2, -3, 3, -4, 4, -5, 5, \ldots \}
\]

**Proof.** Define \( f: \mathbb{N} \to \mathbb{Z} \) by:

\[
f(n) = \begin{cases} 
\frac{n}{2} & \text{if } n \text{ is even} \\
-\frac{n+1}{2} & \text{if } n \text{ is odd}
\end{cases}
\]

We check that \( f \) is bijective.

**Injectivity.** Take \( m, n \in \mathbb{N} \) such that \( f(m) = f(n) \). If \( f(m) = f(n) \)
is \( \geq 0 \), then \( m \) and \( n \) are both even and:

\[
f(n) = \frac{n}{2} = \frac{m}{2} \Rightarrow n = m.
\]

If \( f(m) = f(n) < 0 \), then \( m \) and \( n \) are both odd and:

\[
-\frac{n+1}{2} = -\frac{m+1}{2} \Rightarrow n = m.
\]

So \( f \) is injective.
Subjectivity: Take any \( a \in \mathbb{Z} \). If \( a > 0 \), then \( 2a \in \mathbb{N} \) and satisfies \( f(2a) = a \). If \( a < 0 \), then \( -2a > 0 \) and \( -2a + 1 \in \mathbb{N} \) which satisfies \( f(-2a + 1) = a \).

So, \( f \) is surjective.

More surprising is the following:

The \( \mathbb{Q}^+ \) is denumerable.

It's a little tricky to prove this formally, so let's just give the idea.

First, let's explain why the set \( \mathbb{Q}^+ \) of positive rational numbers is denumerable.

Any \( x \in \mathbb{Q}^+ \) can be written \( x = \frac{a}{b} \) with \( a, b \in \mathbb{N} \).

Consider the square:

\[
\begin{array}{cccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \cdots \\
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \cdots \\
\frac{4}{1} & \frac{4}{2} & \frac{4}{3} & \frac{4}{4} & \cdots \\
\end{array}
\]

How do we write all the elements of this square in a list?

By using the diagonals?

By listing them according to the diagonals and deleting repetitions, we obtain:

\[ \mathbb{Q}^+ = \frac{1}{1}, \frac{1}{2}, \frac{3}{1}, \frac{2}{2}, \frac{3}{2}, \frac{1}{3}, \frac{4}{1}, \frac{3}{3}, \frac{2}{4}, \cdots \]

Since every \( \frac{a}{b} \) with \( a, b \in \mathbb{N} \) will be on one of these diagonals, this does list all elements of \( \mathbb{Q}^+ \).

So we have a bijection \( f : \mathbb{N} \to \mathbb{Q}^+ \).

Now define \( g : \mathbb{Z} \to \mathbb{Q} \) by:

\[ g(n) = \begin{cases} \frac{n}{1} & \text{if } n > 0 \\ \frac{-n}{1} & \text{if } n < 0 \end{cases} \]
\[
\begin{align*}
g(a) &= \begin{cases} 
  f(a) & \text{if } a > 0 \\
  0 & \text{if } a = 0 \\
  -f(-a) & \text{if } a < 0 
\end{cases} 
\end{align*}
\]

Since \( f \) is a bijection, it is easy to check that \( g \) is a bijection, so
\[
|E| = |Z|
\]

But \( |Z| = \aleph_0 \) by the previous theorem, so
\[
|E| = \aleph_0.
\]

Rule: To give a formal proof, you would have to define the function \( f \). It's tricky to deal with the "delete repetitions" part.