Theorem: Let \( \sim \) be an equivalence relation on a set \( A \), and let \( a, b \in A \). Then the following are equivalent:

1. \( a \sim b \)
2. \( \overline{\{a\}} \cap \overline{\{b\}} = \emptyset \)
3. \( \overline{\{a\}} = \overline{\{b\}} \)

Proof:

1 \( \Rightarrow \) 2: Assume that \( a \sim b \). Then \( b \in \overline{\{a\}} \).
Since \( b \in \overline{\{b\}} \) by reflexivity, we have \( b \in \overline{\{a\}} \cap \overline{\{b\}} \).
Hence \( \overline{\{a\}} \cap \overline{\{b\}} = \emptyset \).

2 \( \Rightarrow \) 3: Assume that \( \overline{\{a\}} \cap \overline{\{b\}} = \emptyset \). Thus \( \exists c \in \overline{\{a\}} \cap \overline{\{b\}} \).
So \( a \sim c \) and \( b \sim c \).

First, \( a \sim c \Rightarrow c \sim a \) by symmetry.

Then \( b \sim c \) and \( c \sim a \) \( \Rightarrow b \sim a \) by transitivity, and \( b \sim a \) and \( a \sim x \) \( \Rightarrow b \sim x \) by transitivity again.

So \( x \in \overline{\{b\}} \) and \( \overline{\{a\}} \subseteq \overline{\{b\}} \).
The proof that \( \overline{\{b\}} \subseteq \overline{\{a\}} \) is similar.

3 \( \Rightarrow \) 1: Assume that \( \overline{\{a\}} = \overline{\{b\}} \). By reflexivity, \( b \in \overline{\{b\}} \).
Since \( \overline{\{a\}} = \overline{\{b\}} \), \( b \in \overline{\{a\}} \) and \( a \sim b \).

In particular, the theorem says that if two equivalence classes \( \overline{\{a\}} \) and \( \overline{\{b\}} \) are not equal, then they are disjoint.

Note also that any \( a \in A \) lies in some equivalence class, namely \( a \in \overline{\{a\}} \). These two facts motivate the following definition.
Define let \( A \) be a set. A partition of \( A \) is a collection \( \{ A_i : i \in I \} \) of subsets \( A_i \subseteq A \) satisfying the following:

1. \( \bigcup_{i \in I} A_i = A \)
2. If \( A_i \neq A_j \), then \( A_i \cap A_j = \emptyset \).

Idea: “Chop up \( A \) into non-overlapping pieces.”

Ex. Are the following partitions?

* Is \( \{ [n, n+1] : n \in \mathbb{Z} \} \) a partition of \( \mathbb{R} \)?
  No: \( [0, 1] \cup [1, 2] \) and \( [0, 1] \cap [1, 2] \neq \emptyset \).

* Is \( \{ [m, m+1] : m \in \mathbb{Z} \} \) a partition of \( \mathbb{R} \)?
  Yes: every \( x \in \mathbb{R} \) satisfies \( n \leq x \leq n+1 \) for some \( n \in \mathbb{Z} \), and \( [n, n+1] \neq [m, m+1] \) \( \Rightarrow m \neq n \Rightarrow m < n \Rightarrow [n, n+1] \cap [m, m+1] = \emptyset \).

* Is \( \{ (n, n+1) : n \in \mathbb{Z} \} \) a partition of \( \mathbb{R} \)?
  No. For example, \( 0 \notin (0, 1) \) since \( 0 < 0 < 1 \).

* Let \( A = \mathbb{R}^2 \) and for each \( r > 0 \), let \( A_r = \{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 = r^2 \} \).
  Is \( \{ A_r : r > 0 \} \) a partition of \( \mathbb{R}^2 \)? Yes.

* Is \( \{ 11, 23, 33, 63, 43, 53, 33 \} a partition of \( \{ 1, 2, 3, 4, 5, 6 \} \)?
  Yes.
Let \( n \in \mathbb{N} \). For each \( 0 \leq m < n \), let
\[
A_m = \{ a \in \mathbb{Z} : a \equiv m \pmod{n} \}.
\]
Is \( \{ A_0, A_1, \ldots, A_{n-1} \} \) a partition of \( \mathbb{Z} \)?

Yes. By the division algorithm, for every \( a \in \mathbb{Z} \),
there is a unique \( 0 \leq b < n \) such that \( a \in [b]_n \).

Let \( A \) be a set.
1. If \( \sim \) is an equivalence relation on \( A \), then the equivalence classes \( [a] \) form a partition of \( A \).
2. Let \( \mathcal{A} = \{ A_i : i \in I \} \) be a partition of \( A \). Define a relation \( \sim \) on \( A \) by
   \[
a \sim b \iff \exists i \in I \text{ such that } a \in A_i \text{ and } b \in A_i.
   \]
   Then \( \sim \) is an equivalence relation.

Proof 1. For any \( a \in A \), we have \( a \in [a] \) by reflexivity of \( \sim \), so the union of all equivalence classes is \( A \).
If \( [a], [b] \in A/\sim \) with \( [a] \neq [b] \), the diagram from the beginning of the lecture yields \( [a] \cap [b] = \emptyset \).
Thus, the equivalence classes \( [a] \sim \) form a partition of \( A \).

2. Reflexive: Take \( a \in A \). Since \( \mathcal{A} = \{ A_i : i \in I \} \) is a partition of \( A \), \( \cup A_i = A \), and \( \exists i \in I \) such that \( a \in A_i \).
   Thus \( a \sim a \).

Symmetric: Take \( a, b \in A \) such that \( a \sim b \). Then \( \exists i \in I \) such that \( a \in A_i \) and \( b \in A_i \).
   Thus \( b \sim a \).

Transitive: Take \( a, b, c \in A \) such that \( a \sim b \) and \( b \sim c \).
   Then \( \exists i, j \in I \) such that \( a \in A_i \) and \( b, c \in A_j \).
   Thus \( b \in A_j \cap A_i \), \( A_i \neq \emptyset \). Thus \( A_i = A_j \).
   Since \( \mathcal{A} = \{ A_i : i \in I \} \) is a partition, \( A_i \cap A_j = \emptyset \Rightarrow A_i = A_j \).
   Thus \( a \in A_i = A_j \), and \( a \sim c \).
We can use this to define seemingly strange equivalence relations.

**Example (Möbius band)** Let \( A = [0,2] \times [0,1] \subset \mathbb{R}^2 \).

Define a partition of \( A \) by:

- If \((x,y) \in A\) has \( 0 < x < 2 \), put \((x,y)\) in a set by itself \( \{(x,y)\}\).
- For any \( y \in [0,1] \) be put \((0,y)\) and \((2,1-y)\) together in a set of 2 elements \( \{(0,y), (2,1-y)\}\).

We can visualize \( A/\sim \) by “gluing” elements in the same equivalence class.