Recall that if for each $n \in \mathbb{Z}_{\geq 0}$, we have a set $A_n$, then
\[ \bigcap_{n=0}^\infty A_n = \{ x : A_n \ni x \text{ for all } n \in \mathbb{Z}_{\geq 0} \} \]

Let $C_0 = [0, 1]$. We let $C_1 = C_0 \setminus \text{(middle third)}$, more precisely
\[ C_1 = [0, \frac{1}{3}] \cup \left( \frac{2}{3}, 1 \right] \]

Let $C_2 = C_1 \setminus \text{(middle third of both intervals)}$, i.e.
\[ C_2 = [0, \frac{1}{6}] \cup \left( \frac{1}{6}, \frac{1}{3} \right] \cup \left( \frac{2}{3}, \frac{5}{6} \right] \cup \left( \frac{5}{6}, 1 \right] \]

We continue in this way. For each $n \in \mathbb{Z}_{\geq 0}$, we let
\[ C_n \text{ be the set obtained by deleting the middle third of every interval in } C_{n-1} \]

Say for example $n = 3$, then
\[ C_3 = [0, \frac{1}{9}] \cup \left[ \frac{2}{9}, \frac{7}{9} \right] \cup \left[ \frac{8}{9}, 1 \right] \]
\[ \cup \left[ \frac{2}{3} \cdot 2, \frac{7}{9} \right] \cup \left[ \frac{1}{3} \cdot 2, \frac{5}{9} \right] \cup \left[ \frac{8}{9}, 2 \cdot \frac{5}{9} \right] \cup \cdots \]
\[ \cup \left[ \frac{2}{3} \cdot 2^n, 1 \right] \]

Define. The Cantor set is
\[ C = \bigcap_{n=0}^\infty C_n. \]
This set has a number of interesting/strange properties.
It's both "big" and "small" at the same time.
It is also the 1st example of a fractal: it is "self-similar."

Smallness

The Cantor set has length 0.
Indeed,
\[ \text{length}(C) = 1, \quad \text{length}(C_1) = \frac{2}{3}, \quad \text{length}(C_2) = \frac{4}{9} \]
Since \( C_{n+1} \) deletes the middle third interval from \( C_n \)
\[ \text{length}(C_{n+1}) = \frac{2}{3} \text{length}(C_n) \]
Thus
\[ \text{length}(C_n) = \left( \frac{2}{3} \right)^n \quad \forall n \in \mathbb{Z}_{\geq 0} \]
\[ C = C_n \quad \forall n \geq 0 \] we have
\[ \text{length}(C) < \left( \frac{2}{3} \right)^n \quad \forall n \in \mathbb{Z}_{\geq 0} \]
But \( \lim_{n \to \infty} \left( \frac{2}{3} \right)^n = 0 \), \therefore
\[ \text{length}(C) = 0. \]

Bigness

\( C \) is infinite. To see this, note that for any \( n \in \mathbb{Z}_{\geq 0} \),
the endpoints of the endpoints of the intervals of \( C_n \) do not
get removed in any subsequent step since we only delete middle
third intervals. Thus these endpoints remain in \( C \).

In particular, since \( x_n \) is an endpoint of \( C_n \), we see
that \( x_n \in C \) for all \( n \in \mathbb{Z}_{\geq 0} \).

In fact, we can find a bijection
\[ f: C \to \left( \mathbb{R} \setminus \mathbb{Q} \right)^\ast \]
(We’ll come back to this later)
Self-similarity

For $A \subseteq \mathbb{R}$, let $A_3 = \{\frac{a}{3} \in \mathbb{Q} : a \in A\}$.

Equation $[0; 1] = \left[0, \frac{1}{3}\right]

Note that $x \in A_{\frac{1}{3}} \iff 3x \in A$.

And let $A_{\frac{1}{3} + \frac{2}{3}} = \left\{\frac{a}{3} + \frac{2}{3} \in \mathbb{R} : a \in A\right\}

Equation $\left[\frac{2}{3}, 1\right] = \left[\frac{2}{3}, 1\right]

Note that $x \in A_{\frac{1}{3} + \frac{2}{3}} \iff 3x - 2 \in A$.

Then for any $n \in \mathbb{N}$

$$C_n = \frac{C_{n-1}}{3} \cup \left(\frac{C_{n-1} + 2}{3}\right)$$

Then

$$x \in C \iff x \in C_n \quad \forall n \geq 0

\iff x \in C_n \quad \forall n \geq 1 \quad \text{since } C_n \subseteq C \quad \forall n \in \mathbb{N}

\iff x \in \frac{C_{n-1}}{3} \cup \left(\frac{C_{n-1} + 2}{3}\right) \quad \forall n \geq 1

\iff x \in \frac{C}{3} \cup \left(C + \frac{2}{3}\right)

So

$$C = \frac{C}{3} \cup \left(C + \frac{2}{3}\right).$$

This is $C$ is made up of 2 shrunken copies of itself.
Such self-similarity is a basic property of fractals.

Ternary representation

Any $x \in [0, 1]$ has a decimal expansion

$$x = \frac{a_1}{10} + \frac{a_2}{100} + \frac{a_3}{1000} + \cdots$$

What does this mean precisely?

$$x = \sum_{n=1}^{\infty} a_n 10^{-n}$$

For any \( \epsilon \in \mathbb{Z}_{+} \) and any \( x \in [0, 1] \),

\[
x = \sum_{n=1}^{\infty} a_n \epsilon^{-n} \quad \text{with} \quad a_n \in \{0, 1, \ldots, \epsilon-1\}
\]

**Idea:** Divide the interval into \( \epsilon \) equal length intervals,

\[
[0,1] = [0, \frac{1}{\epsilon}] \cup \left[ \frac{1}{\epsilon}, \frac{2}{\epsilon} \right] \cup \cdots \cup \left[ \frac{\epsilon-1}{\epsilon}, 1 \right]
\]

If \( x \in \left[ \frac{a}{\epsilon}, \frac{a+1}{\epsilon} \right) \), then \( a_1 = 0 \), if \( x \in \left[ \frac{a}{\epsilon}, \frac{a+1}{\epsilon} \right) \), then \( a_1 = 1 \), etc.

Then divide this interval into \( \epsilon \) equal length intervals.

If \( x \) is in the first, \( a_2 = 0 \), if \( x \) is in the second, \( a_2 = 1 \), etc. Continue.

If \( \epsilon = 3 \), we get the ternary representation of \( x \in [0, 1] \),

let's write

\[
x = \sum_{n=1}^{\infty} a_n \epsilon^{-n}
\]

with \( a_n \in \{0, 1, 2\} \)

**Eg:** \([0, 1/2) = \frac{1}{3} + \frac{0}{9} + \frac{2}{27} = \frac{11}{27}

\([0.02020202\ldots] = \sum_{n=1}^{\infty} a_n \epsilon^{-n} \quad \text{with} \quad a_n = \{0, 1, 0, 2, 1, 0, \ldots\} \quad \text{if not odd}

= \sum_{n=1}^{\infty} \frac{2^0}{3^n}

= 2 \sum_{n=1}^{\infty} \frac{1}{9^n}

= 2 \left( \frac{\frac{1}{9}}{1 - \frac{1}{9}} \right) = \frac{1}{4}

**The Cantor set is the set of \( x \in [0, 1] \) whose ternary expansion containing only 0s and 2s.**

**Eg:** \( \frac{1}{4} \in C \), But \( \frac{1}{2} \) is not the endpoint of any interval in any \( C_n \).

**Proof:** We prove first that for any \( n \in \mathbb{Z}_{+} \), \( C_n \) is the set
If $x \in [0,1]$ whose first $n$ digits in the ternary expansion are 0s or 2s. We prove this by induction on $n \in \mathbb{Z}_{\geq 0}$.

Base case When $n=0$, there is nothing to check, so the base case is automatically true.

Inductive step Fix $n \in \mathbb{Z}_{\geq 0}$ and assume that $C_n$ is the set of $x \in [0,1]$ whose first $n$ digits are 0s and 2s. Let $x \in C_{n+1}$. We have

$$C_{n+1} = \frac{C_n}{3} \cup \left( \frac{C_n}{3} + \frac{2}{3} \right)$$

So $x \in C_n$ or $x \in \frac{C_n}{3} + \frac{2}{3}$. If $x \in \frac{C_n}{3}$, then $\exists y \in C_n$ such that $x = \frac{y}{3}$.

By our inductive hypothesis,

$$y = [0, a_1, a_2, \ldots, a_n, 3^i] \quad \text{with } a_i \in \{0, 2\} \text{ for } 0 \leq i \leq n$$

The

$$x = \frac{y}{3} = [0, a_1, a_2, \ldots, a_n]$$

has its first $n+1$ digits, $0, a_1, a_2, \ldots, a_n \in \{0, 2\}$.

If $x \in \frac{C_n}{3} + \frac{2}{3}$, from previous case, $y \in C_n$ such that $x = \frac{y}{3} + \frac{2}{3}$.

Writing $y$ as before-p

$$x = \frac{y}{3} + \frac{2}{3} = [0, a_1, a_2, \ldots, a_n, 3^i]$$

has its first $n+1$ digits, $2, a_1, a_2, \ldots, a_n \in \{0, 2\}$.

This proves the inductive step.

By induction, for all $n \in \mathbb{Z}_{\geq 0}$, $C_n$ is the set where first $n$ digits are 0 or 2.

Since $C = \bigcup C_n$, we conclude that $C$ is the set of $x \in [0,1]$ such that all digits in the ternary expansion are 0s or 2s. \qed