Topics: Finite sets and bijections
      Composition of functions

Lemma 1 (Pigeonhole principle) Let A and B be finite sets. If \( \exists \) a bijection \( f: A \to B \), then \(|A| \leq |B|\).

Proof: Let \( \eta = |A| \) and write
      \[ A = \{ a_1, a_2, \ldots, a_\eta \} \]
      Let \( f: A \to B \) be an injection. Since \( f \) is an injection, the elements \( f(a_1), f(a_2), \ldots, f(a_\eta) \) are distinct elements of \( B \).
      Thus \(|B| > \eta\). \( \Box \)

Lemma 2: Let A and B be finite sets. If \( \exists \) a surjection \( g: B \to A \), then \(|A| \geq |B|\).

Proof: Let \( \mu = |B| \) and write \( B = \{ b_1, b_2, \ldots, b_\mu \} \). Let \( f: A \to B \) be a surjection. Then for every \( 1 \leq i \leq \mu \), \( \exists a_i \in A \) such that \( f(a_i) = b_i \). These \( a_1, a_2, \ldots, a_\mu \) are all distinct by the definition of a function. So \(|A| \geq \mu\). \( \Box \)

Then Let A and B be finite sets. The following are equivalent:
1. \(|A| \leq |B|\)
2. \( \exists \) an injection \( f: A \to B \)
3. \( \exists \) a surjection \( g: B \to A \)

Remark: The theorem is asserting that \( 1 \iff 2 \) and \( 1 \iff 3 \). This may seem like you have to prove 6 different assertions, but you don't. For example, it's enough to prove \( 1 \iff 2 \) and \( 2 \iff 3 \).
because \( 2 \Rightarrow 3 \) then follows. In fact, the most economical

thing is to prove something like

\[ 1 \Rightarrow 2, \; 2 \Rightarrow 3, \; \text{and} \; 3 \Rightarrow 1. \]

**Proof.** The lemmas show that \( 2 \Rightarrow 1 \) and \( 3 \Rightarrow 1 \).

Set it suffices to prove \( 1 \Rightarrow 2 \) and \( 1 \Rightarrow 3 \).

Let \( m = \lvert A \rvert \) and \( n = \lvert B \rvert \), and define

\[ A = \{a_1, a_2, \ldots, a_m\} \text{ and } B = \{b_1, b_2, \ldots, b_n\} \]

(1) \( \Rightarrow 2 \): Assume \( m \leq n \). Define \( f : A \to B \) by

\[ f(a_i) = b_i \text{ for } 1 \leq i \leq m. \]

Since \( b_1, b_2, \ldots, b_n \) are all distinct, this is injective.

(1) \( \Rightarrow 3 \): Define \( g : B \to A \) by \( g(b_i) = a_i \text{ for } 1 \leq i \leq n. \)

This function is surjective.

Cor. For finite sets \( A \) and \( B \),

\[ \lvert A \rvert = \lvert B \rvert \iff \exists \text{ a bijection } f : A \to B. \]

**Proof.** If \( f : A \to B \) is a bijection, then the

Lemma implies that \( \lvert A \rvert = \lvert B \rvert \).

Now assume \( \lvert A \rvert = \lvert B \rvert \). By the theorem,

\[ \exists \text{ an injection } f : A \to B. \]

Then \( \lvert A \rvert = \lvert f(A) \rvert \).

So \( \lvert B \rvert = \lvert A \rvert = \lvert f(A) \rvert \). Since \( B \) is a finite

set and \( f(A) \subseteq B \), this implies \( f(A) = B \), i.e.

\[ f \text{ is surjective.} \]

\( \Box \)
Composition of Functions

Def. Let A, B, and C be sets and let f: A → B and g: B → C be functions. The composition of f and g, written $g \circ f: A \rightarrow C$ is defined by

$$(g \circ f)(a) = g(f(a)).$$

Example 1:
Let $f: \mathbb{R} \setminus \{-3\} \rightarrow \mathbb{R} \setminus \{0\}$ be given by $f(x) = x^2 + 1$. Let $g: \mathbb{R} \rightarrow \mathbb{R}$ be $g(x) = \frac{1}{x}$. Then $g \circ f: \mathbb{R} \setminus \{-3\} \rightarrow \mathbb{R}$ given by

$$(g \circ f)(x) = \frac{1}{(x^2 + 1)} = \frac{1}{(x+1)^2} = (x+1)^{-2}.$$

Note that we can't form the composite $f \circ g$ since the codomain of $f$ is not the domain of $g$. In particular, if we try $(f \circ g)(-1) = f((-1)^3) = f(-1)$, it doesn't make sense since $-1$ is not in the domain of $f$.

Example 2:
Let $f: \mathbb{N} \rightarrow \mathbb{N}$ be defined by

$$f(x) = \begin{cases} 3x + 1 & \text{if } x \text{ is odd} \\ \frac{x}{2} & \text{if } x \text{ is even.} \end{cases}$$

Then can permute $f$. For example,

$$(f \circ f)(5) = f(f(5)) = f(16) = 8.$$

Can also permute $f \circ f$, and $f \circ (f \circ (f \circ f))$, etc.
For example:
\[
f_0(f_1(f_2(f_3(f_4(5)))) = f(f(f(f(f(5)))))) = f(f(f(f(16)))) = f(f(f(8))) = f(f(4)) = f(2) = 1
\]

Aside: It is believed that for any \( n \in \mathbb{N} \), \( f(n) \) such that \((f\circ f\circ f\circ f)(n) = 1\) we don't know how to prove this.

Let \( f: A \to B \) and \( g: B \to C \) be functions.

1. If \( f \) and \( g \) are both injective, then so is \( g \circ f \).
2. If \( f \) and \( g \) are both surjective, then so is \( g \circ f \).

In particular, if \( f \) and \( g \) are both bijective, then so is \( g \circ f \).

**Proof of 1:** Let \( a, a_1, a_2 \in A \) be such that \((g \circ f)(a) = (g \circ f)(a_1)\).

We want to show \( a = a_2 \).

\((g \circ f)(a) = (g \circ f)(a_1) \Rightarrow g(f(a)) = g(f(a_1))\) by definition.

\( \Rightarrow f(a) = f(a_1)\) since \( g \) is injective.

\( \Rightarrow a = a_2\) since \( f \) is injective. \( \square \)

This theorem has a partial converse:

**Theorem:** Let \( f: A \to B \) and \( g: B \to C \) be functions.

1. If \( g \circ f \) is injective, then \( f \) is injective.
2. If \( g \circ f \) is surjective, then \( g \) is surjective.

**Proof of 2:** Take \( c \in C \). We want to show \( \exists b \in B \) such that \( g(b) = c \).
Since \( g \circ f \) is surjective, we know there is \( a \in A \) such that \( (g \circ f)(a) = C \). Then \( P(a) \circ b = P(a) \in B \), so \( g(b) = g(P(a)) = C \). 

**Rule:** In part 1, \( g \) does not need to be injective. In part 2, \( f \) does not need to be surjective.

**Eg:** We saw previously that

\[ f: [0, \infty) \to \mathbb{R} \text{ given by } f(x) = x^2 \text{ is injective but not surjective} \]

\[ g: \mathbb{R} \to [0, \infty) \text{ given by } g(x) = x^2 \text{ is surjective but not injective} \]

The composite

\[ g \circ f: [0, \infty) \to [0, \infty) \text{ is given by } (g \circ f)(x) = x^4 \]

is bijective. (Check yourself!)