(1) For each of the following collections of subsets \( \{ A_r : r \in \mathbb{R} \} \), determine whether or not it determines a partition of \( \mathbb{R}^2 \). Justify your answers, and sketch a few of the subsets \( A_r \).

(a) \( A_r = \{(x, y) \in \mathbb{R}^2 : y = 2x + r\} \).
(b) \( A_r = \{(x, y) \in \mathbb{R}^2 : y = (x - r)^2\} \).
(c) \( A_r = \{(x, y) \in \mathbb{R}^2 : xy = r\} \).

(2) Let \( n \in \mathbb{N} \).

(a) Prove that the operation \( \cdot \) on \( \mathbb{Z}_n \) is well-defined.

For the rest of this question, we write \( \cdot \) for \( \cdot \mod n \) and + for \( + \mod n \), to make the notation a little easier.

(b) Prove by induction (on \( m \)) that for any \( m \in \mathbb{N} \) and any \( a \in \mathbb{Z} \), we have \([a^m] = [a]^m\). (Where, as you probably guessed, \([a]^m\) means \( [a] \cdot [a] \cdot \ldots \cdot [a] \) \( m \) times.

(c) Find all possible values of \( n \in \mathbb{N} \) such that \([5]^m \cdot [9] + [7] = [2]^6\) is true.

(3) Define a relation \( \sim \) on \( \mathbb{Z} \times \mathbb{N} \) by \((a, b) \sim (c, d) \iff ad = bc\).

(a) Prove, without using division, that \( \sim \) is an equivalence relation.

(b) Prove that the function \( f : \mathbb{Z} \times \mathbb{N} \mod \sim \to \mathbb{Q} \) given by \( f([ (a, b) ]) = \frac{a}{b} \) is well-defined and bijective.

(4) Define a partition of the sphere \( S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\} \) into two element subsets of the form \( \{(x, y, z), (-x, -y, -z)\} \), and let \( \sim \) denote the resulting equivalence relation.

(a) Explain why the function \( g : s^2 \mod \sim \to \mathbb{R} \) given by \( g([ (x, y, z) ]) = xyz \) is not well-defined.

(b) Prove that the function \( f : s^2 \mod \sim \to \mathbb{R}^3 \) given by \( f([ (x, y, z) ]) = (yz, zx, xy) \) is well-defined. (The image of this function is a well-known surface called the Roman surface or Steiner surface. You can find pictures of it online.)

(c) For the function \( f \) in part (b), find \( f^{-1}(\{(0, 0, 0)\}) \).

Extra

The following questions will not be graded and do not need to be turned in. They are only here in case you would like extra practice/challenge.

(5) Let \( \sim \) be the equivalence relation on \( \mathbb{Z} \times \mathbb{N} \) of question (3). Define operations \( \oplus \) and \( \otimes \) on \( \mathbb{Z} \times \mathbb{N} \mod \sim \) such that the function \( f \) of (3) part (b) satisfies
\[
\begin{align*}
f([ (a, b) ] \oplus [ (c, d) ]) &= f([ (a, b) ]) + f([ (c, d) ]) \\
f([ (a, b) ] \otimes [ (c, d) ]) &= f([ (a, b) ]) \cdot f([ (c, d) ]) 
\end{align*}
\]
for all \([ (a, b) ], [ (c, d) ] \in \mathbb{Z} \times \mathbb{N} \mod \sim \). Prove directly that \( \oplus \) and \( \otimes \) are well-defined.