MATH 347 – HOMEWORK 5

(1) For \( n \in \mathbb{N} \), find a formula for \( \sum_{j=1}^{n} \frac{1}{j(j+1)} \) (try plugging in a few small values for \( n \) and see if you notice a pattern). Prove that your formula is correct for all \( n \in \mathbb{N} \).

(2) Determine the set of natural numbers \( n \) such that \( 2^n \geq n^2 \). Prove that your guess is correct.

(3) For each integer \( n \geq 0 \), define integers \( c_n \) as follows
\[
c_0 = 3, \quad c_1 = 2, \quad c_2 = 6, \quad \text{and} \quad c_n = 2c_{n-1} + c_{n-2} - 2c_{n-3} \quad \text{for} \quad n \geq 3.
\]
Prove that \( c_n = 1 + (-1)^n + 2^n \) for all integers \( n \geq 0 \).

(4) Recall that on a previous worksheet we proved that if \( p \) is a prime number and \( a, b \) are integers such that \( p \mid (ab) \), then \( p \mid a \) or \( p \mid b \). Use this together with induction to prove the following generalization:
Let \( p \) be a prime number. For any \( n \in \mathbb{N} \) and any \( a_1, \ldots, a_n \in \mathbb{Z} \), if \( p \mid (a_1 \cdot a_2 \cdots a_n) \), then \( p \mid a_i \) for some \( 1 \leq i \leq n \).

(5) In class we used strong induction to prove that any integer \( n \geq 2 \) is either a prime or the product of primes. Give a different proof of this theorem that uses a proof by contradiction and the fact that \( \mathbb{N} \) is well-ordered.

(6) Let \( \mathbb{Z}_{\geq 0} \) be the set of nonnegative integers. In this exercise, we’ll use the fact that \( \mathbb{Z}_{\geq 0} \) is well-ordered to prove the division algorithm (yay!):

**Theorem** (Division algorithm). For any integers \( a \) and \( b \) with \( b > 0 \), there are unique integers \( q \) and \( r \) such that \( a = bq + r \) and \( 0 \leq r < b \).

(a) Let \( S = \{ s \in \mathbb{Z}_{\geq 0} : s = a - bq \text{ for some } q \in \mathbb{Z} \} \). Show that \( S \neq \emptyset \).
(b) By part (a) and the fact that \( \mathbb{Z}_{\geq 0} \) is well-ordered, the set \( S \) has a minimal element \( r \). Prove that \( 0 \leq r < b \).
(c) Finally, prove the uniqueness: Let \( q_1, r_1, q_2, r_2 \in \mathbb{Z} \) be such that \( a = bq_i + r_i \) and \( 0 \leq r_i < b \) for \( i = 1, 2 \). Prove that \( q_1 = q_2 \) and \( r_1 = r_2 \).

**Extra**

The following questions will not be graded and do not need to be turned in. They are only here in case you would like extra practice/challenge.

(7) Prove the following super important theorem:

**Theorem** (The fundamental theorem of arithmetic). For any integer \( n \geq 2 \), there is \( k \in \mathbb{N} \) and prime numbers \( p_1, \ldots, p_k \) such that \( n = p_1 \cdots p_k \). Moreover, this expression is unique up to order of the primes.

(We already showed that \( n \) can be written as the product of primes. The point now is to show the uniqueness. To that end, you need to prove that if you have
\[
p_1 \cdots p_k = q_1 \cdots q_m
\]
then \( k = m \) and that after reordering \( q_1, \ldots, q_m \), if necessary, we have \( p_i = q_i \) for each \( 1 \leq i \leq k = m \).

Due date: Wednesday, October 9 at the beginning of class.