1. Recall that a function \( f : A \to B \) is invertible if there is a function \( g : B \to A \) such that for all \( a \in A \) and \( b \in B \), we have \( f(a) = b \iff g(b) = a \).

(a) (5 points) Let \( f : [0, 1] \to [1, 2] \) be given by \( f(x) = x^2 + 1 \) for \( x \in [0, 1] \). Prove that \( f \) is invertible by verifying the definition, i.e. find a function \( g : [1, 2] \to [0, 1] \) and show that it satisfies the definition above.

**Solution.** Define \( g : [1, 2] \to [0, 1] \) by \( g(y) = \sqrt{y - 1} \). Then for \( x \in [0, 1] \) and \( y \in [1, 2] \), we have
\[
f(x) = y \iff x^2 + 1 = y \iff x^2 = y - 1 \iff x = \sqrt{y - 1} \iff x = g(y),
\]
where in the second to last “\( \iff \)” we have used the fact that \( x \geq 0 \) and \( y \geq 1 \).

(b) (1 point) State a property that is equivalent to a function \( f : A \to B \) being invertible (but that is not the definition).

**Solution.** A function \( f : A \to B \) is invertible if and only if it is bijective.

(c) Explain briefly why the following functions are not invertible.

i. (2 points) \( f : [-1, 1] \to [1, 2] \) given by \( f(x) = x^2 + 1 \).

**Solution.** This function is not injective since \( f(-1) = 2 = f(1) \).

ii. (2 points) \( f : [0, 1] \to [0, 2] \) given by \( f(x) = x^2 + 1 \).

**Solution.** This function is not surjective since \( x^2 + 1 \geq 1 \) for every \( x \in [0, 1] \), so \( 0 \in [0, 2] \) is not in the image of \( f \).

2. Let \( A \) be a set and let \( \sim \) be an equivalence relation on \( A \). Recall that for any \( a \in A \), the equivalence class of \( a \) is the set \( [a] = \{ x \in A : a \sim x \} \).

(a) (2 points) Explain why \( a \in [a] \). Deduce that every element of \( A \) belongs to some equivalence class.

**Solution.** Since \( \sim \) is reflexive, \( a \sim a \) which implies that \( a \in [a] \).
(b) (6 points) Using only the definitions of equivalence relations and equivalence classes, prove that for any \( a, b \in A \), if \([a] \cap [b] \neq \emptyset\), then \([a] = [b]\).

**Solution.** By assumption, there is \( c \in [a] \cap [b] \), so \( a \sim c \) and \( b \sim c \), by symmetry. Then by transitivity, \( a \sim b \). By symmetry again, we also have that \( b \sim a \).

Take \( x \in [a] \), so \( a \sim x \). Then \( b \sim a \) and \( a \sim x \), so \( b \sim x \) by transitivity. This shows that \([a] \subseteq [b]\).

Take \( x \in [b] \), so \( b \sim x \). Then \( a \sim b \) and \( b \sim x \), so \( a \sim x \) by transitivity. This shows that \([b] \subseteq [a]\).

This proves that \([a] = [b]\). \(\square\)

(c) (2 points) Fill in the blank: By parts (a) and (b), we deduce that the collection of equivalence classes form a **partition** of \( A \).

3. Define the relation \( \sim \) on \( \mathbb{R}^2 \) by \((x, y) \sim (u, v) \iff x - u = y - v\).

(a) (5 points) Prove that \( \sim \) is an equivalence relation.

**Solution.** For any \((x, y) \in \mathbb{R}^2\), we have \((x, y) \sim (x, y)\) since \(x - x = 0 = y - y\). So \( \sim \) is reflexive.

Now take \((x, y), (u, v) \in \mathbb{R}^2\) such that \((x, y) \sim (u, v)\). Then \(x - u = y - v\), and multiplying both sides by \(-1\), we have \(u - x = v - y\), so \((u, v) \sim (x, y)\). This shows that \( \sim \) is symmetric.

Finally, we check transitivity. Take \((x, y), (u, v), (s, t) \in \mathbb{R}^2\) such that \((x, y) \sim (u, v)\) and \((u, v) \sim (s, t)\). Then \(x - u = y - v\) and \(u - s = v - t\). Adding these two equations, we get \(x - s = y - t\), so \((x, y) \sim (s, t)\).

We have shown that \( \sim \) is reflexive, symmetric, and transitive, so it is an equivalence relation. \(\square\)

(b) (5 points) Is the function \( f: \mathbb{R}^2/\sim \to \mathbb{R} \) given by \( f([((x, y)]) = (x - y)^3 \) well-defined? Justify your answer.

**Solution.** It is well-defined. We check this directly.

Take \((x, y), (u, v) \in \mathbb{R}^2\) such that \([((x, y)]) = [((u, v)]\). Then \((x, y) \sim (u, v)\) and \(x - u = y - v\). Rearranging this equation, we have \(x - y = u - v\). Then

\[
f([((x, y)]) = (x - y)^3 = (u - v)^3 = f([((u, v)]).
\]

This shows that \( f \) is well-defined. \(\square\)
4. Let \( A \) and \( B \) be sets.

(a) \( (2 \text{ points}) \) Define what \(|A| \leq |B|\) means.

**Solution.** \(|A| \leq |B|\) means that there is an injection \( f: A \to B \).

(b) \( (2 \text{ points}) \) Define what \(|A| = |B|\) means.

**Solution.** \(|A| = |B|\) means that there is a bijection \( f: A \to B \).

5. \( (6 \text{ points}) \) For each of the following, state whether or not the set is denumerable, has cardinality equal to the continuum, or has cardinality strictly greater than the continuum. No justification is required.

(a) The set of prime numbers.

**Solution.** Denumerable.

(b) The power set of the Cantor set.

**Solution.** Strictly greater than the continuum.

(c) \( \mathbb{Q}^2 \).

**Solution.** Denumerable.

(d) The power set of \( \mathbb{Z} \).

**Solution.** Continuum.

(e) \([0,1] \cup [2,3]\).

**Solution.** Continuum.

(f) \( \mathbb{R} \setminus \mathbb{Q} \).

**Solution.** Continuum.

6. \( (5 \text{ points}) \) Let \( \mathbb{Q}^+ \) be the set of positive rational numbers. Show that \( \mathbb{Q}^+ \) is denumerable. (You may do this by showing that the elements of \( \mathbb{Q}^+ \) can be listed.)

**Solution.** Consider the array

\[
\begin{array}{cccccc}
\frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \cdots \\
\frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \cdots \\
\frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \cdots \\
\frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
\]
whose entry in the $i$th row and $j$th column is $\frac{i}{j}$, for each $i, j \in \mathbb{N}$. Note that every element $r \in \mathbb{Q}^+$ appears in this array, since we can write $r = \frac{i}{j}$ for $i, j \in \mathbb{N}$. Each diagonal with $i + j$ constant contains only finitely many entries, so we can list all the entries of this table by first listing the elements $\frac{i}{j}$ with $i + j = 2$, i.e. $\frac{1}{1}$, then the elements with $i + j = 3$, i.e. $\frac{2}{1}, \frac{1}{2}$, then the elements with $i + j = 4$, etc. Deleting fractions that define a rational number that has previously appeared in our list, we obtain a listing

$$\mathbb{Q}^+ = \left\{ \frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{3}{1}, \frac{2}{2}, \frac{1}{3}, \frac{4}{1} \right\}.$$ 

This shows that $\mathbb{Q}^+$ is denumerable, as this listing is equivalent to the bijection $f : \mathbb{N} \to \mathbb{Q}^+$ given by $f(n) =$ the $n$th element in our list. □

7. (5 points) Let $\{0,1\}^\mathbb{N}$ be the set of all sequences $(a_k)_{k=1}^\infty$ with $a_k \in \{0,1\}$ for all $k \in \mathbb{N}$, and let $\mathbb{N}^\mathbb{N}$ be the set of all sequences $(n_k)_{k=1}^\infty$ with $n_k \in \mathbb{N}$ for all $k \in \mathbb{N}$. Prove or disprove: $|\{0,1\}^\mathbb{N}| = |\mathbb{N}^\mathbb{N}|$.

Solution. We will prove that $|\{0,1\}^\mathbb{N}| = |\mathbb{N}^\mathbb{N}|$.

First, define the function $f : \{0,1\}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ by $f((a_k)_{k=1}^\infty) = (a_k + 1)_{k=1}^\infty$. For any $(a_k)_{k=1}^\infty, (b_k)_{k=1}^\infty \in \{0,1\}^\mathbb{N}$, we have

$$f((a_k)_{k=1}^\infty) = f((b_k)_{k=1}^\infty) \implies (a_k + 1)_{k=1}^\infty = (b_k + 1)_{k=1}^\infty \implies a_k + 1 = b_k + 1 \text{ for all } k \in \mathbb{N} \implies a_k = b_k \text{ for all } k \in \mathbb{N} \implies (a_k)_{k=1}^\infty = (b_k)_{k=1}^\infty.$$ 

This shows that $f$ is injective.

Now define a function $g : \mathbb{N}^\mathbb{N} \to \{0,1\}^\mathbb{N}$ by defining $g((n_k)_{k=1}^\infty)$ to be the sequence that starts with $n_1$ many 0s, followed by $n_2$ many 1s, followed by $n_3$ many 0s, etc. We give a more formal definition of $g$ below and prove that it is injective.\(^1\) Granting this for the moment, we have injective functions $f : \{0,1\}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ and $g : \mathbb{N}^\mathbb{N} \to \{0,1\}^\mathbb{N}$, so by the Cantor–Schroeder–Bernstein Theorem, we deduce that $|\{0,1\}^\mathbb{N}| = |\mathbb{N}^\mathbb{N}|$.

We check that $g$ is injective. Formally, we set $g((n_k)_{k=1}^\infty) = (a_k)_{k=1}^\infty$ where we define $(a_k)_{k=1}^\infty$ as follows: for each $k \in \mathbb{N}$, let $l \in \mathbb{N}$ be such that $\sum_{j=1}^{l-1} n_j < k \leq \sum_{j=1}^l n_j$,\(^2\) and set $a_k = (-1)^l + 1$. Now take $(n_k)_{k=1}^\infty, (m_k)_{k=1}^\infty \in \mathbb{N}^\mathbb{N}$ and write $g((n_k)_{k=1}^\infty) = (a_k)_{k=1}^\infty$ and $g((m_k)_{k=1}^\infty) = (b_k)_{k=1}^\infty$. If $(n_k)_{k=1}^\infty \neq (m_k)_{k=1}^\infty$, then $\{ k \in \mathbb{N} : n_k \neq m_k \} \neq \emptyset$, so by the well-ordering principle, it has a least element $s$. Without loss of generality, we can assume that $n_s < m_s$, and we let $r = \sum_{j=1}^{s-1} n_j$. Since $n_j = m_j$ for all $j < s$ and $n_s < m_s$, we have $\sum_{j=1}^s n_j < r + 1 \leq \sum_{j=1}^{s+1} n_j$ and $\sum_{j=1}^{s-1} m_j < r + 1 \leq \sum_{j=1}^s m_j$. By the definition of $g$, this implies that

$$a_{r+1} = (-1)^{s+1} + 1 \neq (-1)^s + 1 = b_{r+1},$$

so $(a_k)_{k=1}^\infty \neq (b_k)_{k=1}^\infty$. This shows that $g$ is injective. □

\(^1\)When grading, I accepted any rough justification for the injectivity of $g$ (or a similar function). In these solutions, I include a formal proof of this injectivity for the sake of completeness.

\(^2\)When $l = 1$, the sum $\sum_{j=1}^{l-1} n_j$ is interpreted as being 0.
**Idea for an alternate solution.** Another choice for an injective function $g : \mathbb{N}^\mathbb{N} \rightarrow \{0,1\}^\mathbb{N}$ is to let $g((n_k)_{k=1}^\infty)$ be the sequence starting with $n_1$ many 0s, followed by a single 1, followed by $n_2$ many 0s, followed by a single 1, etc.