1. (a) Let $P$ and $Q$ be propositions. Compute the truth tables for the given propositions.

   i. (3 points) $(\neg P) \lor Q$

   Solution.

   \[
   \begin{array}{cc|c|c}
   P & Q & \neg P & (\neg P) \lor Q \\
   \hline
   T & T & F & T \\
   T & F & F & F \\
   F & T & T & T \\
   F & F & T & T \\
   \end{array}
   \]

   ii. (3 points) $P \implies (P \land Q)$

   Solution.

   \[
   \begin{array}{cc|c|c|c}
   P & Q & P \land Q & P \implies (P \land Q) \\
   \hline
   T & T & T & T \\
   T & F & F & F \\
   F & T & F & T \\
   F & F & T & T \\
   \end{array}
   \]

   (b) (1 point) Is either of the propositions from part (a) a tautology?

   Solution. No, neither is a tautology (neither is always true).

   (c) (1 point) Is either of the propositions from part (a) a contradiction?

   Solution. No, neither is a contradiction (neither is always false).

   (d) (1 point) Are the two the propositions from part (a) logically equivalent to one another?

   Solution. Yes, as they have the same truth values.
2. (a) (2 points) State the converse of the proposition:

For positive integers \(a\) and \(b\), if \(a = b\) then \(a \mid b\) and \(b \mid a\).

**Solution.** For positive integers \(a\) and \(b\), if \(a \mid b\) and \(b \mid a\), then \(a = b\).

(b) (3 points) State the contrapositive of the proposition:

For a prime number \(p\) and \(a, b \in \mathbb{Z}\), if \(p \nmid a\) and \(p \nmid b\), then \(p \nmid (ab)\).

**Solution.** For a prime number \(p\) and \(a, b \in \mathbb{Z}\), if \(p \mid (ab)\), then \(p \mid a\) or \(p \mid b\).

(c) (4 points) Without using words of negation (e.g. "not"), negate the following proposition:

There exist \(x, y \in \mathbb{R}\) with \(x < y\) such that for all \(r \in \mathbb{Q}\), \(r \leq x\) or \(r \geq y\).

**Solution.** For all \(x, y \in \mathbb{R}\) with \(x < y\), there exists \(r \in \mathbb{Q}\) such that \(x < r < y\).

3. For each of the following, prove the statement or give a counterexample.

(a) (5 points) For all positive real numbers \(x, y\), we have \(x/y + y/x \leq 2xy\).

**Solution.** This is false. For example, consider \(x = y = \frac{1}{2}\). Then

\[
\frac{x}{y} + \frac{y}{x} = 1 + 1 = 2,
\]

and

\[
2xy = 2 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}.
\]

So \(\frac{x}{y} + \frac{y}{x} > 2xy\) when \(x = y = \frac{1}{2}\). □

(b) (5 points) For \(a, b, c, x, y \in \mathbb{Z}\), if \(c \nmid (ax + by)\), then \(c \nmid a\) or \(c \nmid b\).

**Solution.** We prove this by contrapositive, i.e. we prove that for any \(a, b, c, x, y \in \mathbb{Z}\), if \(c \mid a\) and \(c \mid b\), then \(c \mid (ax + by)\).

Since \(c \mid a\) and \(c \mid b\), there are \(m, n \in \mathbb{Z}\) such that \(a = cm\) and \(b = cn\). Then

\[
ax + by = (cm)x + (cn)y = c(mx + ny),
\]

so \(c \mid (ax + by)\). □
4. (4 points) State the **division algorithm**.

**Solution.** For any integer $a$ and any positive integer $b$, there exist unique integers $q$ and $r$ such that $a = bq + r$ and $0 \leq r < b$.

5. (a) (4 points) Use the Euclidean algorithm to find the greatest common divisor of 98 and 77.

**Solution.** The Euclidean algorithm yields:

\[
\begin{align*}
98 &= 77 \cdot 1 + 21 \\
77 &= 21 \cdot 3 + 14 \\
21 &= 14 \cdot 1 + 7 \\
14 &= 7 \cdot 2.
\end{align*}
\]

So $\gcd(98, 77) = 7$.

(b) (4 points) Find $x, y \in \mathbb{Z}$ such that $98x + 77y = \gcd(98, 77)$.

**Solution.** Reversing the Euclidean algorithm, we have:

\[
\begin{align*}
7 &= 21 - 14 \cdot 1 \\
&= 21 - (77 - 21 \cdot 3) \cdot 1 \\
&= 21 \cdot 4 - 77 \cdot 1 \\
&= (98 - 77 \cdot 1) \cdot 4 - 77 \cdot 1 \\
&= 98 \cdot 4 - 77 \cdot 5
\end{align*}
\]

So $x = 4$ and $y = -5$ satisfies $98x + 77y = \gcd(98, 77)$.

6. (5 points) Let $n$ be a positive integer and let $a, b, c, d$ be integers. Prove that if $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $ac \equiv bd \pmod{n}$.

**Solution.** Since $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, we have that $n \mid (a - b)$ and $n \mid (c - d)$. So there are $k, m \in \mathbb{Z}$ such that $a - b = nk$ and $c - d = nm$, equivalently $a = b + nk$ and $c = d + nm$. Then

\[
ac = (b + nk)(d + nm) = bd + bnm + dnk + n^2km = bd + n(bm + dk + nkm),
\]

so $n \mid (ac - bd)$, which implies that $ac \equiv bd \pmod{n}$.
7. (5 points) Let \( p \) be an odd prime and let \( a \) be an integer with \( 1 \leq a \leq p - 1 \). Prove that there is an integer \( 1 \leq b \leq p - 1 \) such that \( b \neq a \) and \( b^2 \equiv a^2 \pmod{p} \).

**Solution.** Let \( b = p - a \). Since \( 1 \leq a \leq p - 1 \), we have \( 1 \leq b \leq p - 1 \).

Next, we claim that \( b \neq a \). Indeed, if \( b = a \), then \( p - a = a \) and \( 2a = p \). But \( p \) is a prime and \( p \neq 2 \), so \( 2 \nmid p \). Thus \( b \neq a \).

Finally, we check that

\[
b^2 = (p - a)^2 \\
\equiv (-a)^2 \pmod{p} \\
\equiv a^2 \pmod{p}.
\]

\( \square \)