Math 347 Final Exam Practice  
Fall Semester 2019

1. State the converse and the contrapositive of each of the following propositions.
   (a) If \( c \mid a \) and \( c \mid b \), then \( c \mid (ax + by) \) for all \( x, y \in \mathbb{Z} \).

   **Solution.**

   Converse: If \( c \mid (ax + by) \) for all \( x, y \in \mathbb{Z} \), then \( c \mid a \) and \( c \mid b \).

   Contrapositive: If there exist \( x, y \in \mathbb{Z} \) such that \( c \nmid (ax + by) \), then \( c \nmid a \) or \( c \nmid b \).

   (b) If \( f : A \to B \) and \( g : B \to C \) are both injective, then \( g \circ f : A \to C \) is injective.

   **Solution.**

   Converse: If \( g \circ f : A \to C \) is injective, then \( f : A \to B \) and \( g : B \to C \) are both injective.

   Contrapositive: If \( g \circ f : A \to C \) is not injective, then \( f : A \to B \) is not injective or \( g : B \to C \) is not injective.

2. State the negation of each of the following propositions.
   (a) Every element of the set \( A \) is either an element of the set \( B \) or an element of the set \( C \).

   **Solution.** There is an element of \( A \) that is not an element of \( B \) and is not an element of \( C \).

   (b) For every \( \epsilon > 0 \), there is \( N \in \mathbb{N} \) such that for all \( n, m \geq N \), we have \( |x_n - x_m| < \epsilon \).

   **Solution.** There exists \( \epsilon > 0 \) such that for all \( N \in \mathbb{N} \), there exist \( n, m \geq N \), such that \( |x_n - x_m| \geq \epsilon \).

3. Find all \( x, y \in \mathbb{Z} \) such that \( 57x + 96y = 6 \) or prove that none exist.

   **Solution.** We first perform the Euclidean algorithm to find \( \gcd(96, 57) \).

   \[
   \begin{align*}
   96 &= 57 \cdot 1 + 39 \\
   57 &= 39 \cdot 1 + 18 \\
   39 &= 18 \cdot 2 + 3 \\
   18 &= 3 \cdot 6 + 0
   \end{align*}
   \]
So \( \gcd(96, 57) = 3 \). Reversing our work, we find that
\[
3 = 39 - 2 \cdot 18 \\
= 39 - 2(57 - 39) \\
= 3 \cdot 39 - 2 \cdot 57 \\
= 3(96 - 57) - 2 \cdot 57 \\
= 3 \cdot 96 - 5 \cdot 57.
\]

Since \( 57(-5) + 96(3) = 3 \), we have \( 57(-10) + 96(6) = 6 \). Then the set of all \( x, y \in \mathbb{Z} \) such that \( 57x + 96y = 6 \) is given by
\[
\begin{align*}
x &= -10 + \frac{96}{3} k = -10 + 32k \\
y &= 6 - \frac{57}{3} k = 6 - 19k,
\end{align*}
\]
for \( k \in \mathbb{Z} \). □

4. Let \( f: A \rightarrow B \) and \( g: B \rightarrow C \) be functions. For each of the following, determine whether or not the statement is true or false. Justify your answers.

(a) If \( g \circ f \) is injective, then \( f \) is injective.

**Solution.** This is true. Take any \( a_1, a_2 \in A \) such that \( f(a_1) = f(a_2) \). Then \( g \circ f(a_1) = g(f(a_1)) = g(f(a_2)) = g \circ f(a_2) \). By the injectivity of \( g \circ f \), we have \( a_1 = a_2 \). This shows that \( f \) is injective. □

(b) If \( g \circ f \) is injective, then \( g \) is injective.

**Solution.** This is not true in general. For example, let \( \mathbb{R}_{\geq 0} \) be the set of nonnegative real numbers, let \( f: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R} \) be given by \( f(x) = x \), and let \( g: \mathbb{R} \rightarrow \mathbb{R}_{\geq 0} \) be given by \( g(x) = x^2 \). Then \( g \circ f: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \) is given by \( g \circ f(x) = x^2 \) and is injective since \( x_1^2 = x_2^2 \implies x_1 = x_2 \) for all \( x_1, x_2 \geq 0 \). On the other hand, \( g \) is not injective since \( g(-1) = 1 = g(1) \). □

(c) If \( g \circ f \) is surjective, then \( f \) is surjective.

**Solution.** This is not true in general. Let \( f \) and \( g \) be as in (b). Then \( g \circ f \) is surjective, since for any \( y \in \mathbb{R}_{\geq 0}, \sqrt{y} \in \mathbb{R}_{\geq 0} \) and \( g \circ f(\sqrt{y}) = (\sqrt{y})^2 = y \). On the other hand, there is no \( x \in \mathbb{R}_{\geq 0} \) such that \( f(x) = -1 \) since \( f(x) = x \geq 0 \) for all \( x \in \mathbb{R}_{\geq 0} \). □
(d) If $g \circ f$ is surjective, then $g$ is surjective.

**Solution.** This is true. Let $c \in C$. Since $g \circ f$ is surjective, there is $a \in A$ such that $g \circ f(a) = c$. Letting $b = f(a) \in B$, we have $g(b) = g(f(a)) = g \circ f(a) = c$. This shows that $g$ is surjective. \hfill \Box

5. For each $r \in \mathbb{R}$, let $A_r = \{(x, y) \in \mathbb{R}^2 : xy = r\}$.

(a) Find $\bigcup_{r \in \mathbb{R}} A_r$. Justify your answer.

**Solution.** We’ll show that $\bigcup_{r \in \mathbb{R}} A_r = \mathbb{R}^2$. Since $A_r \subseteq \mathbb{R}^2$ for all $r \in \mathbb{R}$, we have $\bigcup_{r \in \mathbb{R}} A_r \subseteq \mathbb{R}^2$. On the other hand, given any $(x, y) \in \mathbb{R}^2$, we have $(x, y) \in A_{xy}$, so there exists $r \in \mathbb{R}$ such that $(x, y) \in A_r$. This shows that $\bigcup_{r \in \mathbb{R}} A_r = \mathbb{R}^2$. \hfill \Box

(b) Is the indexed collection $\{A_r : r \in \mathbb{R}\}$ pairwise disjoint? Justify your answer.

**Solution.** Yes. Take any $r, s \in \mathbb{R}$ and assume that $A_r \cap A_s \neq \emptyset$. Take $(x, y) \in A_r \cap A_s$. Then $(x, y) \in A_r$ implies that $xy = r$ and $(x, y) \in A_s$ implies that $xy = s$. So $r = xy = s$, which shows that the collection is pairwise disjoint. \hfill \Box

(c) Is the indexed collection $\{A_r : r \in \mathbb{R}\}$ a partition of $\mathbb{R}^2$?

**Solution.** Yes, it follows immediately from part (a) and (b) that $\{A_r : r \in \mathbb{R}\}$ satisfies the definition of a partition of $\mathbb{R}^2$. \hfill \Box

6. For all $n \in \mathbb{N}$, $4 \mid (3^{2n} - 5^n)$.

(a) Prove this using induction.

**Solution.** We first prove the base case. When $n = 1$, we have $3^{2n} - 5^n = 3^2 - 5 = 4$ and $4 \mid 4$, so the result is true when $n = 1$.

Now assume that $4 \mid (3^{2n} - 5^n)$ for some $n \in \mathbb{N}$. We have

$$3^{2(n+1)} - 5^{n+1} = 9 \cdot 3^{2n} - 5 \cdot 5^n = 5 \cdot (3^{2n} - 5^n) + 4 \cdot 3^{2n}.$$  

By the inductive hypothesis, $4 \mid (3^{2n} - 5^n)$ and clearly $4 \mid 4 \cdot 3^{2n}$, so $4$ also divides $(3^{2n} - 5^n) + 4 \cdot 3^{2n} = 3^{2(n+1)} - 5^{n+1}$. This proves the inductive step, and the result follows by induction. \hfill \Box

(b) Prove this using modular arithmetic.
Solution. Note that $2^2 = 9 \equiv 1 \pmod{4}$ and $5 \equiv 1 \pmod{4}$. So for any $n \in \mathbb{N}$,

$$3^{2n} - 5^n = 9^n - 5^n \equiv 1^n - 1^n \pmod{4} \equiv 0 \pmod{4}.$$ 

So $4 \mid 3^{2n} - 5^n$. □

7. Determine whether or not the following relations $R$ satisfy each of the properties of reflexivity, symmetry, and transitivity. Justify your answers.

(a) The relation $R$ on $\mathbb{Z}$ given by $aRb \iff ab$ is even.

Solution. This relation is not reflexive. For example, $1 \not R 1$ since $1 \cdot 1 = 1$ is not even.

This relation is symmetric, since for all $a, b \in \mathbb{Z}$, we have $ab = ba$, so $ab$ even $\iff ba$ is even. Hence $aRb \iff bRa$.

This relation is not transitive. For example $1R2$ since $1 \cdot 2 = 2$ is even, and $2R1$ since $2 \cdot 1 = 2$ is even, but $1 R 1$ since $1 \cdot 1 = 1$ is odd. □

(b) The relation $R$ on $\mathcal{P}(\mathbb{N})$ given by $ARB \iff A \cap B = A$.

Solution.

This relation is reflexive since for any $A \subseteq \mathbb{N}$, we have $A \cap A = A$, so $ARA$.

This relation is not symmetric. For example $\emptyset R \{1\}$ since $\emptyset \cap \{1\} = \emptyset$, but $\{1\} R \emptyset$ since $\{1\} \cap \emptyset = \emptyset \neq \{1\}$.

This relation is transitive. Let $A, B, C \subseteq \mathbb{N}$ be such that $ARB$ and $BRC$. Then $ARB$ means that $A \cap B = A$ which implies that $A \subseteq B$. Similarly $BRC$ implies that $B \subseteq C$. Then $A \subseteq C$ and $A \cap C = A$, so $ARC$. □

8. Let $A$ and $B$ be infinite sets. Prove that $A \cup B$ is denumerable if and only if $A$ and $B$ are both denumerable.

Solution. First assume that $A \cup B$ is denumerable, so there is a bijection $f : A \cup B \to \mathbb{N}$. The map $g : A \to A \cup B$ given by $g(a) = a$ is clearly injective, so $f \circ g : A \to \mathbb{N}$ is injective and $|A| \leq \aleph_0$. But then since $A$ is infinite, we must have $|A| = \aleph_0$, i.e. $A$ is denumerable. The proof that $B$ is denumerable is the same.

Now assume that both $A$ and $B$ are denumerable. Since $A$ and $B$ are infinite, $A \cup B$ is infinite as well. So to prove that $A \cup B$ is denumerable, it suffices to show there is an
injection $f : A \cup B \to \mathbb{N}$. Since $A$ and $B$ are denumerable, there are bijections $g : A \to \mathbb{N}$ and $h : B \to \mathbb{N}$. Define $f : A \cup B \to \mathbb{N}$ by

$$f(x) = \begin{cases} 2g(x) & \text{if } x \in A, \\ 2h(x) + 1 & \text{if } x \in B \setminus A. \end{cases}$$

We claim that $f$ is injective. Take $x, y \in A \cup B$ such that $f(x) = f(y)$. If $f(x) = f(y)$ is even, then by the definition of $f$, we see that $x, y \in A$ and $f(x) = g(x) = g(y) = f(y)$. Since $g$ is injective, $x = y$. If $f(x) = f(y)$ is odd, then by the definition of $f$, we see that $x, y \in B$ and $f(x) = h(x) = h(y) = f(y)$. Since $h$ is injective, $x = y$ in this case as well. This completes the proof. \[\square\]

**Remark.** A less formal but still okay justification of the implication

$$A \text{ and } B \text{ denumerable } \implies A \cup B \text{ is denumerable}$$

is as follows.

Since $A$ and $B$ are denumerable, we have listings $A = \{a_1, a_2, a_3, \ldots, \}$ and $B = \{b_1, b_2, b_3, \ldots, \}$. The consider the listing

$$\{a_1, b_1, a_2, b_2, a_3, b_3, \ldots, \}$$

where we alternate elements from $A$ and from $B$. After deleting any repetitions, we obtain a listing

$$A \cup B = \{x_1, x_2, x_3, \ldots, \}$$

and the function $f : \mathbb{N} \to A \cup B$ given by $f(n) = x_n$ is a bijection. \[\square\]

9. (a) Define what it means for a sequence of real numbers $(x_n)_{n=1}^{\infty}$ to have a **limit** $L \in \mathbb{R}$.

**Solution.** The sequence $(x_n)_{n=1}^{\infty}$ has **limit** $L$ if for all $\epsilon > 0$, there is $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - L| < \epsilon$.

(b) Use the definition to prove that $\lim_{n \to \infty} \left(2 + \frac{7}{(n+1)^3}\right) = 2$.

**Solution.** Fix $\epsilon > 0$. By the Archimedean principle, there is $N \in \mathbb{N}$ such that $N \geq \sqrt[3]{\frac{7}{\epsilon}} - 1$. Then for any $n \geq N$, we have

$$\left| 2 + \frac{7}{(n+1)^3} - 2 \right| = \frac{7}{(n+1)^3} \leq \frac{7}{(N+1)^3} < \frac{7}{\left(\sqrt[3]{\frac{7}{\epsilon}} - 1 + 1\right)^3} = \frac{7}{\epsilon} = \epsilon.$$  \[\square\]
10. Using the definition, prove that if \((x_n)_{n=1}^\infty\) and \((y_n)_{n=1}^\infty\) are Cauchy sequences of real numbers, then \((x_n + y_n)_{n=1}^\infty\) is a Cauchy sequence.

**Solution.** Fix \(\epsilon > 0\). Since \((x_n)_{n=1}^\infty\) is Cauchy, there is \(N_1 \in \mathbb{N}\) such that for all \(n, m \geq N_1\), \(|x_n - x_m| < \frac{\epsilon}{2}\). Similarly, there is \(N_2 \in \mathbb{N}\) such that for all \(n, m \geq N_2\), \(|y_n - y_m| < \frac{\epsilon}{2}\). Let \(N = \max\{N_1, N_2\}\). Then for all \(n, m \geq N\), we have

\[
|(x_n + y_n) - (x_m + y_m)| = |(x_n - x_m) + (y_n - y_m)| \leq |x_n - x_m| + |y_n - y_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

This proves that \((x_n + y_n)_{n=1}^\infty\) is Cauchy. \(\square\)

11. Let \((x_k)_{k=1}^\infty\) be a sequence of real numbers.

(a) Define what it means for the infinite series \(\sum_{k=1}^\infty x_k\) to converge.

**Solution.** The infinite series \(\sum_{k=1}^\infty x_k\) converges if the sequence of partial sums \((s_n)_{n=1}^\infty\) converges, where \(s_n = \sum_{k=1}^n x_k\) for every \(n \in \mathbb{N}\).

(b) Prove that if \(\sum_{k=1}^\infty x_k\) converges, then \(\lim_{k \to \infty} x_k = 0\).

**Solution.** For each \(n \in \mathbb{N}\), let \(s_n = \sum_{k=1}^n x_k\). For convenience, set \(s_0 = 0\). Then for any \(k \in \mathbb{N}\), we have \(x_k = s_k - s_{k-1}\). Since \(\sum_{k=1}^\infty s_k\) converges, the sequence \((s_n)_{n=1}^\infty\) converges; let \(L\) be its limit. Then \((s_{n-1})_{n=1}^\infty\) also has limit \(L\) as it is just a shift of the sequence \((s_n)_{n=1}^\infty\). Then by limit laws, \((x_k)_{k=1}^\infty\) is convergent and

\[
\lim_{k \to \infty} x_k = \lim_{k \to \infty} (s_k - s_{k-1}) = \lim_{k \to \infty} s_k - \lim_{k \to \infty} s_{k-1} = L - L = 0.
\]

\(\square\)

**Alternate solution.** For each \(n \in \mathbb{N}\), let \(s_n = \sum_{k=1}^n x_k\). For convenience, set \(s_0 = 0\). Then for any \(k \in \mathbb{N}\), we have \(x_k = s_k - s_{k-1}\). Since \(\sum_{k=1}^\infty x_k\) converges, the sequence \((s_n)_{n=1}^\infty\) converges, hence is Cauchy. Then for any \(\epsilon > 0\), there is \(N \in \mathbb{N}\) such that for all \(n, m \geq N\), \(|s_n - s_m| < \epsilon\). In particular, for all \(k \geq N + 1\), we then have

\[
|x_k| = |s_k - s_{k-1}| < \epsilon.
\]

This shows that \(\lim_{k \to \infty} x_k = 0\). \(\square\)