Mastery Exam Practice 2 Solutions

(1) Compute the derivative of \( f(x) = \frac{2}{x} \) using the definition of the derivative.

\[
 f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
 = \lim_{h \to 0} \frac{\frac{2}{x+h} - \frac{2}{x}}{h} \\
 = \lim_{h \to 0} \frac{2x - 2(x+h)}{x(x+h)} \cdot \frac{h}{h} \\
 = \lim_{h \to 0} \frac{2h}{x(x+h)} \\
 = \lim_{h \to 0} \frac{-2}{x(x+h)} = -\frac{2}{x^2}
\]

(2) Find an equation of the tangent line to the curve \( y = 3x + \sin(x) \) at the point where \( x = 0 \).

\[
y'(x) = 3 + \cos(x). \ y'(0) = 3 + \cos(0) = 4. \text{ At } x = 0, \ y(0) = 3(0) + \sin(0) = 0, \text{ so the line with slope } 4 \text{ going through } (0, 0) \text{ is given by } y = 4x.
\]

(3) (a) If \( 10^{1.69} = 49 \), what is an approximate value of \( \log_{10}(7) \)?

\[
10^{1.69} = 49 = 7^2. \text{ Thus } 10^{0.845} = 7, \text{ so } \log_{10}(7) = 0.845.
\]

(b) Find \( \tan\left(\frac{\pi}{6}\right) \).

\[
\frac{1}{\sqrt{3}}
\]

(4) Let \( f(x) = 2\ln(x + 1) - 3 \).

(a) Find a formula for \( f^{-1} \).

\[
x = 2\ln(y + 1) - 3 \\
\frac{(x + 3)}{2} = \ln(y + 1) \\
e^{\frac{(x+3)}{2}} = y + 1 \\
y = e^{\frac{(x+3)}{2}} - 1
\]

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(b) What is the domain of \( f^{-1} \)? Explain.
The domain is all real numbers.

(5) Find the derivative of \( f(x) = e^x + \cos(2x) \ln(x) \).
\[ f'(x) = 0 + -2 \sin(2x) \ln(x) + \cos(2x) \frac{1}{x} \]

(6) Find the derivative of \( f(x) = 5^{(x^4 + \cos(x))} + x^6 \).
\[ f'(x) = \ln(5)5^{(x^4 + \cos(x))}(4x^3 - \sin(x)) + 6x^5 \]

(7) Find the derivative of \( f(x) = \frac{e^{15x} + 12x}{\sin(x)} \).
\[ f'(x) = \frac{(15e^{15x} + 12)(\sin(x)) - \cos(x)(e^{15x} + 12)}{\sin^2(x)} \]

(8) Sketch a graph of \( g(x) = 1 + 3 \cos(3x) \). Indicate the scale on both the \( x \) and \( y \) axes.
Check on your calculator.

(9) Suppose that \( f \) is a function with derivative \( f'(x) = (x^2 - 1)e^x \).

(a) Determine the set of points \( x \) on which \( f \) is increasing.
The roots of \( f'(x) \) are 1 and -1. For \( x < -1 \), \( f'(x) > 0 \). For \( x \) in \((-1, 1)\), \( f'(x) < 0 \), and for \( x > 1 \), \( f'(x) > 0 \). Thus \( f \) is increasing on \((-\infty, -1) \cup (1, \infty)\).

(b) Find all the stationary points of the function \( f \).
\[ x = -1, 1 \]

(c) Which of the stationary points are maximum, minimum, or neither? Explain.
By the first derivative test, \( x = -1 \) is a local max and \( x = 1 \) is a local min.

(10) Find the equation of the tangent line to \( \sin(xy) = x + y \) at the point \((0, 0)\).
Implicitly differentiate:
\[ \cos(xy)(y + xy') = 1 + y' \]
\[ \cos(xy)y + \cos(xy)xy' = 1 + y' \]
Solve for $y'$:

\[ y' - \cos(xy)xy' = \cos(xy)y - 1 \]
\[ y'(1 - \cos(xy)x) = \cos(xy)y - 1 \]
\[ y' = \frac{\cos(xy)y - 1}{1 - \cos(xy)x} \]

At $(0, 0)$, $y' = -1/1 = -1$, so the equation of the tangent line is $y = -x$.

(11) If a culture of mold doubles in size every 8 hours and there is 5 grams of mold at noon, find an equation for the grams of mold in terms of hours past noon.

Let $w(t)$ denote the grams of mold in terms of hours past noon. We know that $w(t) = Ae^{kt}$ for some constants $A$ and some constants $t$. Since $w(0) = 5$, we have that $A = 5$. We know that $w(8) = 10$, but $w(8) = 5e^{8k} = 10$, $k = \ln(2)/8$.

Thus $w(t) = 5e^{\ln(2)t}$. 