Math 220 BE1 Practice Exam 3

No calculators, books, cell phones, or notes are to be used during the test.
This exam covers sections 4.2 through 4.9. You must show your work to receive credit. Answer all questions.

1) State in full
(a) The Mean Value Theorem
Suppose that $f$ is continuous on the closed, bounded interval $[a, b]$ and differentiable on the open interval $(a, b)$. Then there is a number $c$ between $a$ and $b$ for which
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

(b) The Extreme Value Theorem
Let $f$ be continuous on the closed, bounded interval $[a, b]$. Then $f$ assumes both a maximum value and a minimum value somewhere on $[a, b]$.

(c) Rolle’s Theorem
Suppose that $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$ and that $f(a) = f(b)$. Then, for some $c$ between $a$ and $b$, $f'(c) = 0$.

(d) The Intermediate Value Theorem
Let $f$ be continuous on the closed, bounded interval $[a, b]$, and let $y$ be any number between $f(a)$ and $f(b)$. Then, for some input $c$ between $a$ and $b$, $f(c) = y$.

2) Find the following limits
(a) $\lim_{x \to \infty} e^{-x} \ln(x)$
$$\lim_{x \to \infty} e^{-x} \ln(x) = \lim_{x \to \infty} \frac{\ln(x)}{e^x}$$
$$= \lim_{x \to \infty} \frac{\frac{1}{x}}{e^x}$$
$$= \lim_{x \to \infty} \frac{1}{xe^x} = 0$$

(b) $\lim_{x \to 0} \frac{1}{\cos(x)}$
$$\lim_{x \to 0} \frac{1}{\cos x} = \frac{1}{\cos(0)} = 1$$
(c) \( \lim_{x \to \pi} \frac{\cos(x - \pi) - 1}{(x - \pi)^3} \).

\[
\lim_{x \to \pi} \frac{\cos(x - \pi) - 1}{x - \pi} = \lim_{x \to \pi} \frac{-\sin(x - \pi)}{3(x - \pi)^2} = \lim_{x \to \pi} \frac{-\cos(x - \pi)}{6(x - \pi)}
\]

We cannot apply L’Hopital’s rule since the numerator goes to 1. From the left-hand side, the limit is \( \infty \), and from the right hand side the limit is \(-\infty\) thus the limit does not exist.

3) (a) A triangle has legs on the positive x-axis and y-axes, and its hypotenuse passes through the point (2, 1). Which such triangle has the smallest area?

We first find the equation of a line passing through (2, 1) as a function of the slope \( m \): \( y - 1 = m(x - 2) \) or equivalently \( y = mx - 2m + 1 \). Notice that \( m < 0 \) if we want the triangle to have its legs on the positive x and y-axes.

Given such a line, the height of the triangle is \(-2m + 1\), and the length of the base is \((2m - 1)/m\). Using the formula for the area of a right triangle we see that \( A(m) = \frac{(2m - 1)(1 - 2m)}{2m} = -2m + 2 - \frac{1}{2m} \). If we set \( \frac{dA}{dm} = 0 \), we see that \(-2 + 5m^{-2} = 0\), which implies that \( m = -0.5 \). The first derivative test indicates that this is a minimum, and it implies that the base is 4 and the height is 2.

*(b) Find all the critical points of \( |x^2 - x| \). Hint: Rewrite the function as a piece-wise function to remove the absolute value.

We first break this up into a piece-wise function:

\[
|x^2 - x| = \begin{cases} 
  x^2 - x & x < 0 \\
  -x^2 + x & 0 \leq x \leq 1 \\
  x^2 - x & x > 1 
\end{cases}
\]

This means that the derivative is given by:

\[
(|x^2 - x|)' = \begin{cases} 
  2x - 1 & x < 0 \\
  -2x + 1 & 0 < x < 1 \\
  2x - 1 & x > 1 
\end{cases}
\]

Thus the derivative is 0 when \( x = .5 \), and undefined when \( x = 0 \) and \( x = 1 \). These are the critical values.

4) Graph the parametric curve given by \( x = 2 \cos(t) \), \( y = t \), \(-\pi \leq t \leq \pi \). Label both axes and indicate the points corresponding to \( t = 0 \) and \( t = -\pi \) on the graph.
5) A lamp positioned on top of a 12 foot pole casts a shadow on a 6 foot man who is walking away from the lamp at the rate of 1 foot per second. How fast is the length of the shadow changing when the man is 8 feet away from the lamp?

Drawing the usual triangle picture, letting \( x(t) \) denote the man’s distance from the lamp and \( s(t) \) the length of the shadow, we see that \( \frac{12}{x+s} = 6s \). Thus \( x = s \), and \( x’ = s’ \). So at all times the shadow is changing at 1 foot per second.

6) (a) Find a third-order Taylor polynomial for \( f(x) = e^x \) expanded around \( x_0 = 0 \).

The derivatives of \( e^x \) are all \( e^x \), and \( e^0 = 1 \), so plugging that into our Taylor Polynomial formula we get that

\[
T_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}
\]

(b) Use the polynomial from part a to estimate the value of \( e \).

Since \( e = e^1 \), we look at \( e^1 \approx T_3(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} = 2 + \frac{2}{3} \).

7) Let \( f(x) = |x - 1| \).

(a) Does \( f \) satisfy the hypothesis of the Intermediate Value Theorem on the interval \([-1, 1]\)? Justify your answer.

Yes, \( f \) is continuous on the closed and bounded interval \([-1, 1]\).

(b) Does \( f \) satisfy the hypothesis of the Extreme Value Theorem on the interval \([-1, 1]\)? Justify your answer.

Yes, \( f \) is continuous on the closed and bounded interval \([-1, 1]\).

(c) Does \( f \) satisfy the conclusion of the Mean Value Theorem on the interval \([-1, 1]\)? Justify your answer.

No: \( \frac{f(1) - f(-1)}{1 - (-1)} = 2/2 = 1 \), and for no value of \( c \) on the interval \((-1, 1)\) is \( f'(c) = 1 \).