Math 525, Spring 2018. Homework 1
Due: Thursday, January 25, 2018

(1) Let X and Y be topological spaces. Suppose A_1 and A_2 are closed subsets of X such that $X = A_1 \cup A_2$. If $f_i : A_i \longrightarrow Y$ are continuous functions that agree on $A_1 \cap A_2$, show that the function

$$f : X \longrightarrow Y, \quad f(x) = \begin{cases} f_1(x) & \text{if } x \in A_1 \\ f_2(x) & \text{if } x \in A_2 \end{cases}$$

is continuous.

(2) Show that a space X is contractible iff every map $f : X \longrightarrow Y$, for arbitrary Y, is nullhomotopic. Similarly, show X is contractible iff every map $f : Y \longrightarrow X$ is nullhomotopic.

(3) Show that $f : X \longrightarrow Y$ is a homotopy equivalence if there exist maps $g, h : Y \longrightarrow X$ such that $fg \cong \mathrm{Id}$ and $hf \cong \mathrm{Id}$. More generally, show that $f : X \longrightarrow Y$ is a homotopy equivalence if there exist $g, h : Y \longrightarrow X$ such that fg and hf are homotopy equivalences.

(4) Show that the number of path components is a homotopy invariant. (That is, show that if X and Y are homotopy equivalent spaces then they have the same number of path components.)

(5) Show that for a space X, the following three conditions are equivalent:
(a) Every map $\mathbb{S}^1 \longrightarrow X$ homotopic to a constant map, with image a point.
(b) Every map $\mathbb{S}^1 \longrightarrow X$ extends to a map $\mathbb{D}^2 \longrightarrow X$.
(c) $\pi_1(X, x_0) = 0$ for all $x_0 \in X$.
Deduce that a space X is simply-connected iff all maps $\mathbb{S}^1 \longrightarrow X$ are homotopic. [In this problem, 'homotopic' means 'homotopic without regard to basepoints.']

(6) Given a space X, a path connected subspace A, and a point $x_0 \in A$, show that the map $\pi_1(A, x_0) \longrightarrow \pi_1(X, x_0)$ induced by the inclusion $A \hookrightarrow X$ is surjective iff every path in X with endpoints in A is homotopic, relative to its endpoints, to a path in A.