(1) Let $X$ and $Y$ be topological spaces. Suppose $A_1$ and $A_2$ are closed subsets of $X$ such that $X = A_1 \cup A_2$. If $f_i : A_i \to Y$ are continuous functions that agree on $A_1 \cap A_2$, show that the function

$$f : X \to Y, \quad f(x) = \begin{cases} f_1(x) & \text{if } x \in A_1 \\ f_2(x) & \text{if } x \in A_2 \end{cases}$$

is continuous.

(2) Show that a space $X$ is contractible iff every map $f : X \to Y$, for arbitrary $Y$, is nullhomotopic. Similarly, show $X$ is contractible iff every map $f : Y \to X$ is nullhomotopic.

(3) Show that $f : X \to Y$ is a homotopy equivalence if there exist maps $g, h : Y \to X$ such that $fg \simeq \text{Id}$ and $hf \simeq \text{Id}$. More generally, show that $f : X \to Y$ is a homotopy equivalence if there exist $g, h : Y \to X$ such that $fg$ and $hf$ are homotopy equivalences.

(4) Show that the number of path components is a homotopy invariant. (That is, show that if $X$ and $Y$ are homotopy equivalent spaces then they have the same number of path components.)

(5) Show that for a space $X$, the following three conditions are equivalent:

(a) Every map $S^1 \to X$ homotopic to a constant map, with image a point.

(b) Every map $S^1 \to X$ extends to a map $D^2 \to X$.

(c) $\pi_1(X, x_0) = 0$ for all $x_0 \in X$.

Deduce that a space $X$ is simply-connected iff all maps $S^1 \to X$ are homotopic. [In this problem, ‘homotopic’ means ‘homotopic without regard to basepoints.’]

(6) Given a space $X$, a path connected subspace $A$, and a point $x_0 \in A$, show that the map $\pi_1(A, x_0) \to \pi_1(X, x_0)$ induced by the inclusion $A \hookrightarrow X$ is surjective iff every path in $X$ with endpoints in $A$ is homotopic, relative to its endpoints, to a path in $A$. 