Math 524 – Linear Analysis on Manifolds
Spring 2012

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Contents

Lecture 1

Differential operators on manifolds

1.1 Smooth manifolds

The m-dimensional Euclidean space $\mathbb{R}^m$, the m-dimensional sphere $\mathbb{S}^m$ and the m-dimensional torus $\mathbb{T}^m = \mathbb{S}^1 \times \ldots \times \mathbb{S}^1$ are all examples of smooth manifolds, the context where our studies will take place. A general manifold is a space constructed from open sets in $\mathbb{R}^m$ by patching them together smoothly.

Formally, M is a **manifold** if it is a paracompact[1] Hausdorff topological space with an 'atlas', i.e., a covering by open sets $\{\mathcal{V}_\alpha\}$ together with homeomorphisms

$$\phi_\alpha : \mathcal{U}_\alpha \longrightarrow \mathcal{V}_\alpha$$

for some open subsets $\{\mathcal{U}_\alpha\}$ of $\mathbb{R}^m$. The integer m is known as the dimension of M and the maps $\phi_\alpha : \mathcal{U}_\alpha \longrightarrow \mathcal{V}_\alpha$ (or sometimes just the sets $\{\mathcal{V}_\alpha\}$) are known as **coordinate charts.** An atlas is smooth if whenever $\mathcal{V}_{\alpha\beta} := \mathcal{V}_\alpha \cap \mathcal{V}_\beta \neq \emptyset$ the map

$$\psi_{\alpha\beta} := \phi_\beta^{-1} \circ \phi_\alpha : \phi_\alpha^{-1}(\mathcal{V}_{\alpha\beta}) \longrightarrow \phi_\beta^{-1}(\mathcal{V}_{\alpha\beta}) \tag{1.1}$$

is a smooth diffeomorphism between the open sets $\phi_\alpha^{-1}(\mathcal{V}_{\alpha\beta}), \phi_\beta^{-1}(\mathcal{V}_{\alpha\beta}) \subseteq \mathbb{R}^m$. (For us smooth will always mean $\mathcal{C}^\infty$.) If the maps $\psi_{\alpha\beta}$ are all $\mathcal{C}^k$-smooth, we say that the atlas is $\mathcal{C}^k$.

If $\phi_\alpha : \mathcal{U}_\alpha \longrightarrow \mathcal{V}_\alpha$ is a coordinate chart, it is typical to write

$$(y_1, \ldots, y_m) = \phi_\alpha^{-1} : \mathcal{V}_\alpha \longrightarrow \mathcal{U}_\alpha$$

and refer to the y_i as *local coordinates* of M on $\mathcal{V}_\alpha$.

[1] A topological space is paracompact if every open cover has a subcover in which each point is covered finitely many times.

Two smooth atlases are compatible (or equivalent) if their union is a smooth atlas. A **smooth structure** on M is an equivalence class of smooth atlases or, equivalently, a maximal smooth atlas on M. A manifold M together with a smooth structure is a **smooth manifold.**

To indicate that a manifold M is m-dimensional we will sometimes write M^m or, more often, we will refer to M as an m**-manifold.**

A more general notion than manifold is that of a **manifold with boundary.** The former is defined in the same way as a manifold, but the local model is

$$\mathbb{R}^m_- = \{(x_1, \dots, x_n) \in \mathbb{R}^m : x_1 \leq 0\}.$$

If M is a smooth manifold with boundary, its boundary ∂M consists of those points in M that are in the image of $\partial \mathbb{R}^m_-$ for some (and hence all) local coordinate charts. It is easy to check that ∂M is itself a smooth manifold (without boundary). The product of two manifolds with boundary is an example of a still more general notion, a *manifold with corners.*

A map between two smooth manifolds (with or without boundary)

$$F : M \longrightarrow M'$$

is smooth if it is smooth in local coordinates. That is, whenever

$$\phi_\alpha : \mathcal{U}_\alpha \longrightarrow \mathcal{V}_\alpha \subseteq M, \quad \phi'_\gamma : \mathcal{U}'_\gamma \longrightarrow \mathcal{V}'_\gamma \subseteq M',$$

are coordinate charts, the map

$$(\phi'_\gamma)^{-1} \circ F \circ \phi_\alpha : \mathcal{U}_\alpha \longrightarrow \mathcal{U}'_\gamma \tag{1.2}$$

is smooth (whenever it is defined). Notice that this definition is equivalent to making coordinate charts diffeomorphisms.

The space of smooth maps between M and M' will be denoted $\mathcal{C}^\infty(M, M')$ and when $M' = \mathbb{R}$, simply by $\mathcal{C}^\infty(M)$. Notice that a map $F \in \mathcal{C}^\infty(M, M')$ naturally induces a pull-back map

$$F^* : \mathcal{C}^\infty(M') \longrightarrow \mathcal{C}^\infty(M), \quad F^*(h)(p) = h(F(p)) \text{ for all } h \in \mathcal{C}^\infty(M'), p \in M.$$

(So we can think of $\mathcal{C}^\infty$ as a contravariant functor on smooth manifolds.)

For us a manifold will mean a connected smooth manifold without boundary, unless otherwise specified. Similarly, unless we specify otherwise, a map between manifolds will be smooth.

A subset $X \subseteq M$ is a **submanifold of** M of dimension k if among the m coordinates on M at each point of X there are k that restrict to give coordinates of X. That is, at each point $p \in X$ there is a coordinate chart of M, $\phi : \mathcal{U} \longrightarrow \mathcal{V}$ with $q \in \mathcal{V}$, whose restriction to $U \cap \mathbb{R}^k \times \{0\}$ maps into X,

$$\phi\big|_{\mathcal{U}\cap\mathbb{R}^k\times\{0\}} : \mathcal{U} \cap \mathbb{R}^k \times \{0\} \longrightarrow \mathcal{V} \cap X.$$

These restricted charts then define a smooth k-manifold structure on X.

1.2 Partitions of unity

A manifold is defined by patching together many copies of $\mathbb{R}^m$. It is often useful to take a construction on $\mathbb{R}^m$ and transfer it to a manifold by carrying it out on each patch and then adding the pieces together.

For these constructions, it is very convenient that every open covering $\mathcal{V}_\alpha$ of a smooth manifold carries a **partition of unity.** That is, a collection ω_i of smooth functions $\omega_i : M \longrightarrow \mathbb{R}$ such that:
i) At each $p \in M$ only finitely many $\omega_i(p)$ are non-zero
ii) Each ω_i is supported inside some $\mathcal{V}_\alpha$
iii) The sum of the ω_i is the constant function equal to one at each point $p \in M$.

One remarkable fact that follows easily from the existence of partitions of unity is that any compact smooth manifold can be identified with a smooth submanifold of $\mathbb{R}^N$ for some N. This is easy to show using a finite atlas with a corresponding partition of unity. Whitney's embedding theorem shows that any smooth manifold M embeds into $\mathbb{R}^{2\dim M+1}$ and a compact smooth manifold into $\mathbb{R}^{2\dim M}$.

1.3 Tangent bundle

We know what a smooth function

$$f : M \longrightarrow \mathbb{R}$$

is, what is its derivative? If γ is a smooth curve on M, that is, a smooth map

$$\gamma : \mathbb{R} \longrightarrow M,$$

then $f \circ \gamma$ is a map from $\mathbb{R}$ to itself so its derivative is unambiguously defined. We can take advantage of this to define the derivative of f at a point $p \in M$ in terms of the curves through that point.

Indeed, let us say that two curves γ_1 and γ_2 are *equivalent at* p if $\gamma_1(0) = \gamma_2(0) = p$ and

$$\frac{\partial}{\partial t}\Big|_{t=0} (\phi^{-1} \circ \gamma_1) = \frac{\partial}{\partial t}\Big|_{t=0} (\phi^{-1} \circ \gamma_2)$$

for some (and hence any) coordinate chart $\phi : \mathcal{U} \longrightarrow \mathcal{V} \subseteq M$ containing p. It is easy to see that this is an equivalence relation and we refer to an equivalence class as a **tangent vector at** $p \in M$. The **tangent space to** M **at** p is the set of all tangent vectors at p and is denoted T_pM.

For the case of Euclidean space, the tangent space to $\mathbb{R}^m$ at any point is another copy of $\mathbb{R}^m$,

$$T_p\mathbb{R}^m = \mathbb{R}^m.$$

The tangent space of the m-sphere is also easy to describe. By viewing $\mathbb{S}^m$ as the unit vectors in $\mathbb{R}^{m+1}$, curves on $\mathbb{S}^m$ are then also curves in $\mathbb{R}^{m+1}$ so we have

$$T_p\mathbb{S}^m \subseteq T_p\mathbb{R}^{m+1} = \mathbb{R}^{m+1},$$

and a tangent vector at p will correspond to a curve on $\mathbb{S}^m$ precisely when it is orthogonal to p as a vector in $\mathbb{R}^{m+1}$, thus

$$T_p\mathbb{S}^m = \{v \in \mathbb{R}^{m+1} : v \cdot p = 0\}.$$

Clearly if γ_1 is equivalent to γ_2 at p then

$$\frac{\partial}{\partial t}\Big|_{t=0} (f \circ \gamma_1) = \frac{\partial}{\partial t}\Big|_{t=0} (f \circ \gamma_2)$$

for all $f : M \longrightarrow \mathbb{R}$, so we can think of the derivative of f at p as a map from equivalence classes of curves through p to equivalence classes of curves through $f(p)$, thus

$$D_pf : T_pM \longrightarrow T_{f(p)}\mathbb{R}.$$

More generally, any smooth map $F : M_1 \longrightarrow M_2$ defines a map, known as its **differential**,

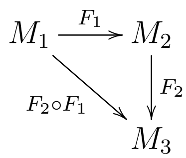
$$D_pF : T_pM_1 \longrightarrow T_{F(p)}M_2$$

by sending the equivalence class of a curve γ on M_1 to the equivalence class of the curve $F \circ \gamma$ on M_2. When the point p is understood, we will omit it from the notation.

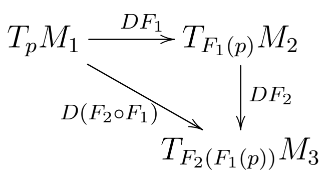
If the differential of a map is injective we say that the map is an **immersion,** if it is surjective we say that it is a **submersion**. If $F : M_1 \longrightarrow M_2$ is an injective immersion which is a homeomorphism between M_1 and $F(M_1)$ then we say that F is an **embedding**. A connected subset $X \subseteq M$ is a submanifold of M precisely when the inclusion map $X \hookrightarrow M$ is an embedding.

If F is both an immersion and a submersion, i.e., if DF is always a linear isomorphism, we say that it is a **local diffeomorphism**. By the inverse function theorem, if F is a local diffeomorphism then every point $p \in M_1$ has a neighborhood $\mathcal{V} \subseteq M_1$ such that $F\big|_{\mathcal{V}}$ is a diffeomorphism onto its image. A **diffeomorphism** is a bijective local diffeomorphism.

The chain rule is the fact that commutative diagrams



of smooth maps between manifolds induce, for each $p \in M_1$, commutative diagrams between the corresponding tangent spaces,



i.e., that $D(F_2 \circ F_1) = DF_2 \circ DF_1$.

Taking differentials of coordinate charts of M, we end up with coordinate charts on

$$TM = \bigcup_{p \in M} T_pM$$

showing that TM, the **tangent bundle** of M, is a smooth manifold of twice the dimension of M. For $\mathbb{R}^m$ the tangent bundle is simply the product

$$T\mathbb{R}^m = \mathbb{R}^m \times \mathbb{R}^m$$

where the first factor is thought of as a point in $\mathbb{R}^m$ and the second as a vector 'based' at that point; the tangent bundle of an open set in $\mathbb{R}^m$ has a very similar description, we merely restrict the first factor to that open set. (We can think of $M \mapsto TM$ together with $F \mapsto DF$ as a covariant functor on smooth manifolds.)

If $\phi : \mathcal{U} \longrightarrow \mathcal{V}$ is a coordinate chart on M and $p \in \mathcal{V} \subseteq M$, then $D_p\phi$ identifies T_pM with $\mathbb{R}^m$. A different choice of coordinate chart gives a different identification of T_pM with $\mathbb{R}^m$, but these identifications differ by an invertible linear map. This means that each T_pM is naturally a vector space of dimension m, which however does not have a natural basis.

Now that we know the smooth structure of TM, note that the differential of a smooth map between manifolds is a smooth map between their tangent bundles.

There is also a natural smooth projection map

$$\pi : TM \longrightarrow M$$

that sends elements of T_pM to p and when $M = \mathbb{R}^m$ is simply the projection onto the first factor. It is typical, for any subset $Z \subseteq M$, to use the notations

$$\pi^{-1}(Z) =: TM\big|_Z =: T_Z M.$$

1.4 Vector bundles

A coordinate chart $\phi : \mathcal{U} \longrightarrow \mathcal{V}$ on a smooth m-manifold M induces a coordinate chart on TM and an identification

$$\pi^{-1}(\mathcal{V}) \xrightarrow{D\phi^{-1}} T\mathcal{U} = \mathcal{U} \times \mathbb{R}^m \xrightarrow{\phi\times\mathrm{id}} \mathcal{V} \times \mathbb{R}^m \tag{1.3}$$

which restricts to each fiber $\pi^{-1}(p)$ to a linear isomorphism. We say that a coordinate chart of M *locally trivializes* TM as a 'vector bundle'.

A **vector bundle** over a manifold M is a locally trivial collection of vector spaces, one for each point in M. Formally a *rank k real vector bundle* consists of a smooth manifold E together with a smooth surjective map $\pi : E \longrightarrow M$ such that the fibers $\pi^{-1}(p)$ form vector spaces over $\mathbb{R}$ and satisfying a local triviality condition: for any point $p \in M$ and small enough neighborhood $\mathcal{V}$ of p there is a diffeomorphism

$$\Phi : \mathcal{V} \times \mathbb{R}^k \longrightarrow \pi^{-1}(\mathcal{V})$$

whose restriction to any $q \in \mathcal{V}$ is a linear isomorphism

$$\Phi\big|_{q\in\mathcal{V}} : \{q\} \times \mathbb{R}^k \longrightarrow \pi^{-1}(q).$$

We refer to the pair $(\mathcal{V}, \Phi)$ as a local trivialization of E.

We will often refer to a rank k real vector bundle as a $\mathbb{R}^k$-bundle. The discussion below of $\mathbb{R}^k$-vector bundles applies equally well to **complex vector bundles,** or $\mathbb{C}^k$-vector bundles, which are the defined by systematically replacing $\mathbb{R}^k$ with $\mathbb{C}^k$ above.

If X is a submanifold of M, and $E \xrightarrow{\pi} M$ is a vector bundle, then

$$E\big|_X \xrightarrow{\pi|_X} X$$

is a vector bundle.

A subset of a vector bundle is a **subbundle** if it is itself a bundle in a consistent fashion. That is, E_1 is a subbundle of E_2 if E_1 and E_2 are both bundles over M, $E_1 \subseteq E_2$, and the projection map $E_1 \longrightarrow M$ is the restriction of the projection map $E_2 \longrightarrow M$.

A **bundle morphism** between two vector bundles

$$E_1 \xrightarrow{\pi_1} M_1, \quad E_2 \xrightarrow{\pi_2} M_2$$

is a map

$$F : E_1 \longrightarrow E_2$$

that sends fibers of π_1 to fibers of π_2 and acts linearly between fibers. Thus for any bundle morphism F there is a map $\overline{F} : M_1 \longrightarrow M_2$ such that the diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{F} & E_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ M_1 & \xrightarrow{\overline{F}} & M_2 \end{array} \tag{1.4}$$

commutes; we say that F covers the map $\overline{F}$. An invertible bundle morphism is called a **bundle isomorphism**.

The trivial $\mathbb{R}^k$-bundle over a manifold M is the manifold $\mathbb{R}^k \times M$ with projection the map onto the second factor. We will use $\underline{\mathbb{R}^k}$ to denote the trivial $\mathbb{R}^k$-bundle. A bundle is trivial if it is isomorphic to $\underline{\mathbb{R}^k}$ for some k. A manifold is said to be **parallelizable** if its tangent bundle is a trivial bundle, e.g., $\mathbb{R}^m$ is parallelizable.

We will show later that $\mathbb{S}^m$ is not parallelizable when m is even. The circle $\mathbb{S}^1$ is parallelizable and hence (by exercise 1) so are all tori $\mathbb{T}^m$. Some interesting facts which we will not show is that any three dimensional manifold is parallelizable and that the only parallelizable spheres are $\mathbb{S}^1$, $\mathbb{S}^3$ and $\mathbb{S}^7$. The fact that these sphere are parallelizable is easy to see by viewing them as the unit vectors in the complex numbers, the quaternions, and the octionians, respectively; the fact that these are the only parallelizable ones is not trivial.

1.5 Pull-back bundles

Any bundle morphism covers a map between manifolds (1.4). There is a sort of converse to this: given a map between manifolds $f : M_1 \longrightarrow M_2$, and a $\mathbb{R}^k$-bundle $E \xrightarrow{\pi} M_2$ the set

$$f^*E = \{(p, v) \subseteq M_1 \times E : f(p) = \pi(v)\}$$

fits into the commutative diagram

$$\begin{array}{ccc} f^*E & \xrightarrow{\widehat{f}} & E \\ \downarrow{\scriptstyle \widehat{\pi}} & & \downarrow{\scriptstyle \pi} \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

where, for any $(p, v) \in f^*E$,

$$p \xleftarrow{\widehat{\pi}} (p, v) \xmapsto{\widehat{f}} v.$$

Moreover, if $(\mathcal{U}, \Phi)$ is a local trivialization of E and $\mathcal{V}$ is a coordinate chart in M_1 such that $f(\mathcal{V}) \subseteq \mathcal{U}$ then

$$f^*E \cap (\mathcal{V} \times \pi^{-1}(\mathcal{U})) = \{(p, (\Phi\big|_{f(p)})^{-1}(x)) : p \in \mathcal{V}, x \in \mathbb{R}^k\}$$

and we can use this to introduce a $\mathcal{C}^\infty$ manifold structure on f^*E such that $f^*E \cap (\mathcal{V} \times \pi^{-1}(\mathcal{U}))$ is diffeomorphic to $\mathcal{V} \times \mathbb{R}^k$ making $f^*E \xrightarrow{\widehat{\pi}} M_1$ a $\mathbb{R}^k$-bundle. The bundle f^*E is called the **pull-back of** E **along** f.

The simplest example of a pull-back bundle is the restriction of a bundle. If X is a submanifold of M, and $E \xrightarrow{\pi} M$ is a vector bundle, then $E\big|_X \xrightarrow{\pi|_X} X$ is (isomorphic to) the pull-back of E along the inclusion map $X \hookrightarrow M$.

We see directly from the definition that the pull-back bundle of a trivial bundle is again trivial, but note that the converse is not true.

Given a smooth map $f : M_1 \longrightarrow M_2$, pull-back along f defines a functor between bundles on Y and bundles on X. Indeed, if $E_1 \longrightarrow Y$ and $E_2 \longrightarrow Y$ are bundles over Y and $F : E_1 \longrightarrow E_2$ is a bundle map between them then

$$f^*F : f^*E_1 \longrightarrow f^*E_2, \quad f^*F(p, v) = (p, F(p)v) \text{ for all } (p, v) \in f^*E_1$$

is a bundle map between f^*E_1 and f^*E_2.

Another useful relation between pull-backs and bundle maps is the following: If $E_1 \xrightarrow{\pi_1} M_1$ and $E_2 \xrightarrow{\pi_2} M_2$ are vector bundles and $F : E_1 \longrightarrow E_2$ is a bundle map so that

$$\begin{array}{ccc} E_1 & \xrightarrow{F} & E_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ M_1 & \xrightarrow{\overline{F}} & M_2 \end{array}$$

then F is equivalent to a bundle map between $E_1 \longrightarrow M_1$ and $\overline{F}^* E_2 \longrightarrow M_1$, namely

$$E_1 \ni v \mapsto (\pi_1(v), F(\pi_1(v))v) \in \overline{F}^* E_2.$$

We will show later (Remark 3) that if $f : M_1 \longrightarrow M_2$ and $g : M_1 \longrightarrow M_2$ are *homotopic smooth maps,* i.e., if there is a smooth map

$$H : [0,1] \times M_1 \longrightarrow M_2 \text{ s.t. } H(0,\cdot) = f(\cdot) \text{ and } H(1,\cdot) = g(\cdot)$$

then $f^*E \cong g^*E$ for any vector bundle $E \longrightarrow M_2$. In particular, this implies that all vector bundles over a contractible space are trivial.

1.6 The cotangent bundle

Every real vector space V has a *dual vector space*

$$V^* = \mathrm{Hom}(V, \mathbb{R})$$

consisting of all the linear maps from V to $\mathbb{R}$. Given a linear map between vector spaces

$$T : V \longrightarrow W$$

there is a dual map between the dual spaces

$$T^* : W^* \longrightarrow V^*, \quad T^*(\omega)(v) = \omega(T(v)) \text{ for all } \omega \in W^*, v \in V.$$

By taking the dual maps of the local trivializations of a vector bundle $E \stackrel{\pi}{\longrightarrow} M$, the manifold

$$E^* = \{(p, \omega) : p \in M, \omega \in (\pi^{-1}(p))^*\}$$

with the obvious projection $E^* \longrightarrow M$ is seen to be a vector bundle, the **dual bundle to** E.

Every vector $v \in \mathbb{R}^k$ defines a dual vector $v^* \in (\mathbb{R}^k)^*$,

$$v^* : \mathbb{R}^k \longrightarrow \mathbb{R}, \quad v^*(w) = v \cdot w,$$

and it is easy to see that the dual vectors of a basis of $\mathbb{R}^k$ form a basis of $(\mathbb{R}^k)^*$ so that in fact $v \mapsto v^*$ is a vector space isomorphism. In particular the dual bundle of an $\mathbb{R}^k$-bundle is again a $\mathbb{R}^k$-bundle.

Given vector bundles $E_1 \longrightarrow M_1$ and $E_2 \longrightarrow M_2$ and $F : E_1 \longrightarrow E_2$ a bundle map, we get a dual bundle map

$$F^* : E_2^* \longrightarrow E_1^*.$$

Thus taking duals defines a contravariant functor on the category of vector bundles.

The dual of a bundle map $F : E_1 \longrightarrow E_2$ between two bundles $E_i \longrightarrow M$ is a map between their duals

$$F^* : E_2^* \longrightarrow E_1^*.$$

Thus taking duals defines a contravariant functor on the category of vector bundles on M. It is easy to see that this functor commutes with pull-back along a smooth map.

The dual bundle to the tangent bundle is the **cotangent bundle**

$$T^*M \longrightarrow M.$$

Remark 1. The assignation $M \mapsto T^*M$ is natural, but is not a functor in the same way that $M \mapsto TM$ is. Rather, a map $F : M_1 \longrightarrow M_2$ between manifolds induces a map $DF : TM_1 \longrightarrow TM_2$ between their tangent bundles which, as explained above, we can interpret as a map $DF : TM_1 \longrightarrow F^*(TM_2)$ between bundles over M_1. Now we can take duals and we get a bundle map $(DF)^* : F^*(TM_2^*) \longrightarrow TM_1^*$.

Thus we can view $M \mapsto T^*M$ as a functor but its target category should consist of vector bundles where a morphism between $E_1 \longrightarrow M_1$ and $E_2 \longrightarrow M_2$ consists of a map $F : M_1 \longrightarrow M_2$ and a bundle map $\Phi : F^*E_2 \longrightarrow E_1$ and composition is

$$(F_1, \Phi_1) \circ (F_2, \Phi_2) = (F_1 \circ F_2, \Phi_1 \circ (F_1^* \Phi_2)).$$

This is called the category of star bundles in [Kolar-Michor-Slovak].

1.7 Operations on vector bundles

We constructed the vector bundle $E^* \longrightarrow M$ from the vector bundle $E \longrightarrow M$ by applying the vector space operation $V \mapsto V^*$ fiber-by-fiber. This works more generally whenever we have a smooth functor of vector spaces.

Let **Vect** denote the category of finite dimensional vector spaces with linear isomorphisms, and let $\mathscr{T}$ be a (covariant) functor on **Vect**. We say that $\mathscr{T}$ is *smooth* if for all V and W in **Vect** the map

$$\mathrm{Hom}(V, W) \longrightarrow \mathrm{Hom}(\mathscr{T}(V), \mathscr{T}(W))$$

is smooth.

Let $E \xrightarrow{\pi} M$ be a vector bundle over M, define

$$\mathscr{T}_p(E) = \mathscr{T}(\pi^{-1}(p)), \quad \mathscr{T}(E) = \bigsqcup_{p \in M} \mathscr{T}_p(E)$$

and let $\mathscr{T}(\pi) : \mathscr{T}(E) \longrightarrow M$ be the map that sends $\mathscr{T}_p(E)$ to p. We claim that $\mathscr{T}(E) \xrightarrow{\mathscr{T}(\pi)} M$ is a smooth vector bundle.

Applying $\mathscr{T}$ fibrewise to a local trivialization of E,

$$\Phi : \mathcal{V} \times \mathbb{R}^k \longrightarrow \pi^{-1}(\mathcal{V}),$$

we obtain a map

$$\mathscr{T}(\Phi) : \mathcal{V} \times \mathscr{T}(\mathbb{R}^k) \longrightarrow \mathscr{T}(\pi)^{-1}(\mathcal{V}),$$

and we can use these maps as coordinate charts on $\mathscr{T}(E)$ to obtain the structure of a smooth manifold. Indeed, we only need to see that if two putative charts overlap then the map

$$(\mathcal{V}_1 \cap \mathcal{V}_2) \times \mathscr{T}(\mathbb{R}^k) \xrightarrow{\mathscr{T}(\Phi_1)^{-1}\mathscr{T}(\Phi_2)} (\mathcal{V}_1 \cap \mathcal{V}_2) \times \mathscr{T}(\mathbb{R}^k)$$

is smooth, and this follows from the smoothness of $\mathscr{T}$.

Thus $\mathscr{T}(E)$ is a smooth manifold, $\mathscr{T}(\pi)$ is a smooth function, and

$$\mathscr{T}(E) \xrightarrow{\mathscr{T}(\pi)} M$$

satisfies the local triviality condition, so we have a new vector bundle over M.

The same procedure works for contravariant functors (meaning that $\mathrm{Hom}(V, W)$ is sent to $\mathrm{Hom}(\mathscr{T}(W), \mathscr{T}(V))$) as well as functors with several variables. Therefore, for instance, we can define for vector bundles $E \longrightarrow M$ and $F \longrightarrow M$ the vector bundles:

i) $\mathrm{Hom}(E, F) \longrightarrow M$ with fiber over p, $\mathrm{Hom}(E_p, F_p)$.

ii) $E \otimes F \longrightarrow M$ with fiber over p, $E_p \otimes F_p$.

iii) $S^2E \longrightarrow M$ whose fiber over p is the space of all symmetric bilinear transformations from $E_p \times E_p$ to $\mathbb{R}$.

iv) $S^kE \longrightarrow M$ whose fiber over p is the space of all symmetric k-linear transformations from E_p^k to $\mathbb{R}$.

v) $\Lambda^kE \longrightarrow M$ whose fiber over p is the space of all antisymmetric k-linear transformations from E_p^k to $\mathbb{R}$.

We also obtain natural isomorphisms of vector bundles from isomorphisms of vector spaces. Thus, for instance, for vector bundles E, F, and G over M, we have[2]

[2]Recall that $\underline{\mathbb{R}}$ is the trivial $\mathbb{R}$-bundle, $\mathbb{R} \times M \longrightarrow M$.

i) $E \oplus F \cong F \oplus E$

ii) $E \otimes \underline{\mathbb{R}} \cong E$

iii) $E \otimes F \cong F \otimes E$

iv) $E \otimes (F \oplus G) \cong (E \otimes F) \oplus (E \otimes G)$

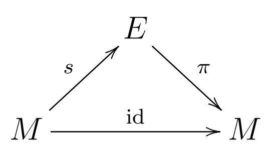
v) $\mathrm{Hom}(E, F) = E^* \otimes F$ (so $E^* \cong \mathrm{Hom}(E, \underline{\mathbb{R}})$)

vi) $\Lambda^k(E \oplus F) \cong \bigoplus_{i+j=k} (\Lambda^i E \otimes \Lambda^j F)$

vii) $(\Lambda^k E)^* = \Lambda^k(E^*)$

1.8 Sections of bundles

A **section** of a vector bundle $E \xrightarrow{\pi} M$ is a smooth map $s : M \longrightarrow E$ such that



commutes. We will denote the space of smooth sections of E by $\mathcal{C}^\infty(M; E)$. This forms a vector space over $\mathbb{R}$ and, as such, it is infinite-dimensional (unless we are in a 'trivial' situation where $(\dim M)(\mathrm{rank} E) = 0$). However it is also a module over $\mathcal{C}^\infty(M)$ (with respect to pointwise multiplication) and here it is finitely generated. For instance, the sections of a trivial bundle form a free $\mathcal{C}^\infty(M)$ module,

$$\mathcal{C}^\infty(M; \underline{\mathbb{R}^n}) = (\mathcal{C}^\infty(M))^n .$$

We also introduce notation for functions and sections with compact support,

$$\mathcal{C}_c^\infty(M), \quad \mathcal{C}_c^\infty(M; E)$$

and point out that the latter is a module over the former. One consequence of the module structure is that we can use a partition of unity on M to decompose a section into a sum of sections supported on trivializations of E.

A section of the tangent bundle is known as a **vector field.** Thus a vector field on M is a smooth function that assigns to each point $p \in M$ a tangent vector in T_pM. A vector field $V \in \mathcal{C}^\infty(M; TM)$ naturally acts on a smooth function by differentiation

$$V : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M), \quad (Vf)(p) = D_p f(V(p)) \text{ for all } f \in \mathcal{C}^\infty(M), p \in M.$$

If $\mathcal{V}_\alpha$ is a coordinate chart on M with local coordinates $y_1, \ldots, y_m$ then we can write

$$V\big|_{\mathcal{V}_\alpha} = \sum_{i=1}^{m} a_i(y_1, \ldots, y_m) \frac{\partial}{\partial y_i}.$$

Indeed, this is just the representation in the induced coordinates of TM with the identification (1.3).

This action of V on $\mathcal{C}^\infty(M)$ is linear and satisfies

$$V(fh) = V(f)h + fV(h), \text{ for all } f, h \in \mathcal{C}^\infty(M),$$

i.e., it is a *derivation* of $\mathcal{C}^\infty(M)$. In fact, it is possible to identify vector fields on M with the derivations of $\mathcal{C}^\infty(M)$. (see e.g., [Lee])

As maps on $\mathcal{C}^\infty(M)$, we can compose vector fields

$$(VW)(f) = V(W(f))$$

but the result is no longer a vector field. However, if we form the commutator of two vector fields,

$$[V, W]f = V(W(f)) - W(V(f))$$

then the result is again a vector field! In fact it is easy to check that this is a derivation on $\mathcal{C}^\infty(M)$. The commutator of vector fields satisfies the 'Jacobi identity'

$$[[U, V], W] + [[V, W], U] + [[W, U], V] = 0$$

and hence the commutator, combined with the $\mathbb{R}$-vector space structure, gives $\mathcal{C}^\infty(M, TM)$ the structure of a *Lie algebra.*

Remark 2. Whereas a smooth map $F : M_1 \longrightarrow M_2$ induces a map $DF : TM_1 \longrightarrow TM_2$, it is not always possible to 'push-forward' a vector field. We say that two vector fields $V \in \mathcal{C}^\infty(M_1; TM_1)$, and $W \in \mathcal{C}^\infty(M_2; TM_2)$ are F-related if the diagram

$$\begin{array}{ccc} TM_1 & \xrightarrow{DF} & TM_2 \\ {\scriptstyle V}\uparrow & & \uparrow{\scriptstyle W} \\ M_1 & \xrightarrow{F} & M_2 \end{array}$$

commutes, i.e., if $DF \circ V = W \circ F$.

If F is a diffeomorphism, then we can push-forward a vector field: the vector fields $V \in \mathcal{C}^\infty(M_1; TM_1)$ and $F_*V = DF \circ V \circ F^{-1} \in \mathcal{C}^\infty(M_2; TM_2)$ are always F-related. Moreover, $F_*[V_1, V_2] = [F_*V_1, F_*V_2]$ for all $V_1, V_2 \in \mathcal{C}^\infty(M_1; TM_1)$, so $M \mapsto \mathcal{C}^\infty(M; TM)$ is a functor from smooth manifolds with diffeomorphisms to Lie algebras and isomorphisms.

1.9 Differential forms

Sections of the cotangent bundle of M, T^*M, are known as **differential 1-forms**. More generally, sections of the k-th exterior power of the cotangent bundle, $\Lambda^k T^*M$, are known as **differential k-forms**. The typical notation for this space is

$$\Omega^k(M) = \mathcal{C}^\infty(M; \Lambda^k T^*M), \quad \Omega^*(M) = \bigoplus_{k=0}^{\dim M} \Omega^k(M)$$

where by convention $\Omega^0(M) = \mathcal{C}^\infty(M)$. A differential form in $\Omega^k(M)$ is said to have degree k.

A differential k-form $\omega \in \Omega^k(M)$ can be evaluated on k-tuples of vector fields,

$$\omega : \mathcal{C}^\infty(M; TM)^k \longrightarrow \mathbb{R},$$

it is linear in each variable and has the property that

$$\omega(V_{\sigma(1)}, \dots, V_{\sigma(k)}) = \operatorname{sign}(\sigma)\omega(V_1, \dots, V_k)$$

for any permutation σ of $\{1, \dots, k\}$.

Differential forms of all degrees $\Omega^*(M)$ form a graded algebra with respect to the **wedge product**

$$\Omega^k(M) \times \Omega^\ell(M) \ni (\omega, \eta) \mapsto \omega \wedge \eta \in \Omega^{k+\ell}(M)$$
$$\omega \wedge \eta(V_1, \dots, V_{k+\ell}) = \frac{1}{k!\ell!} \sum_\sigma \operatorname{sign}(\sigma)\omega(V_{\sigma(1)}, \dots, V_{\sigma(k)})\eta(V_{\sigma(k+1)}, \dots, V_{\sigma(k+\ell)}).$$

This product is graded commutative (so sometimes commutative and sometimes anti-commutative)

$$\omega \in \Omega^k(M), \quad \eta \in \Omega^\ell(M) \implies \omega \wedge \eta = (-1)^{k\ell} \eta \wedge \omega.$$

In particular if $\omega \in \Omega^k(M)$ with k odd, then $\omega \wedge \omega = 0$.

Given a differential k-form, ω, we will write the operator 'wedge product with ω, by

$$\mathfrak{e}_\omega : \Omega^*(M) \longrightarrow \Omega^{*+k}(M), \quad \mathfrak{e}_\omega(\eta) = \omega \wedge \eta, \text{ for all } \eta \in \Omega^*(M).$$

We say that a k-form is elementary if it is the wedge product of k 1-forms. Elementary k-forms are easy to evaluate

$$(\omega_1 \wedge \dots \wedge \omega_k)(V_1, \dots, V_k) = \det(\omega_i(V_j)).$$

Let p be a point in a local coordinate chart $\mathcal{V}$ on M with coordinates $y_1, \dots, y_m$. A basis of T_p^*M dual to the basis $\{\partial_{y_1}, \dots, \partial_{y_m}\}$ of T_pM is given by

$$\{dy^1, \dots, dy^m\} \text{ where } dy^i(\partial_{y_j}) = \delta^i_j.$$

A basis for the space of k-forms in T_p^*M is given by

$$\{dy^{i_1} \wedge \cdots \wedge dy^{i_k} : i_1 < \cdots < i_k\}.$$

Any k-form supported in $\mathcal{V}$ can be written as a sum of elementary k-forms. Using a partition of unity we can write any k-form on M as a sum of elementary k-forms. Notice that anticommutativity of the wedge product implies that all elementary k-forms, and hence all k-forms, of degree greater than the dimension of the manifold vanish.

There is also an **interior product** of a vector field and a differential form, which is simply the contraction of the differential form by the vector field. For every $W \in \mathcal{C}^\infty(M; TM)$, the interior product is a map

$$\mathfrak{i}_W : \Omega^*(M) \longrightarrow \Omega^{*-1}(M)$$
$$(\mathfrak{i}_W\omega)(V_1, \dots, V_{k-1}) = \omega(W, V_1, \dots, V_{k-1}), \text{ for all } \omega \in \Omega^k, \{V_i\} \subseteq \mathcal{C}^\infty(M; TM).$$

We will also use the notation

$$\mathfrak{i}_W\omega = W \lrcorner\, \omega.$$

Notice that antisymmetry of differential forms induces anticommutativity of the interior product in that

$$\mathfrak{i}_V\mathfrak{i}_W = -\mathfrak{i}_W\mathfrak{i}_V$$

for any $V, W \in \mathcal{C}^\infty(M; TM)$. In particular, $(\mathfrak{i}_V)^2 = 0$.

If $F : M_1 \longrightarrow M_2$ is a smooth map between manifolds, we can pull-back differential forms from M_2 to M_1,

$$F^* : \Omega^k(M_2) \longrightarrow \Omega^k(M_1),$$
$$F^*\omega(V_1, \dots, V_k) = \omega(DF(V_1), \dots, DF(V_k)) \text{ for all } \omega \in \Omega^k(M_2), V_i \in \mathcal{C}^\infty(M; TM_1).$$

Pull-back respects the wedge product,

$$F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta, \tag{1.5}$$

and the interior product,

$$F^*(DF(W) \lrcorner\, \omega) = W \lrcorner\, F^*(\omega).$$

In the particular case of $\mathbb{R}^m$, the pull-back of an m-form by a linear map T is

$$T^*(dx^1 \wedge \cdots \wedge dx^m) = (\det T)(dx^1 \wedge \cdots \wedge dx^m), \tag{1.6}$$

directly from the definition of the determinant.

1.10 Orientations and integration

If W is an m-dimensional vector space, and $\{u_1, \dots, u_m\}$, $\{v_1, \dots, v_m\}$ are bases of W, we will say that they *represent the same orientation* if the linear map that sends u_i to v_i has positive determinant. Equivalently, since $\Lambda^m W$ is a one-dimensional vector space, we know that $u_1 \wedge \cdots \wedge u_m$ and $v_1 \wedge \cdots v_m$ differ by multiplication by a non-zero real number and we say that they represent the same orientation if this number is positive. A representative ordered basis or non-zero m-form is said to be an **orientation** of W. The canonical orientation of Euclidean space $\mathbb{R}^m$ corresponds to the canonical ordered basis, $\{e_1, \dots, e_m\}$.

An $\mathbb{R}^k$-vector bundle $E \longrightarrow M$ is **orientable** if we can find a smooth orientation of its fibers, i.e., if there is a nowhere vanishing section of $\Lambda^k E \longrightarrow M$. Notice that the latter is an $\mathbb{R}$-bundle over M, so the existence of a nowhere vanishing section is equivalent to its being trivial. Thus $E \longrightarrow M$ is orientable if and only if $\Lambda^k E \longrightarrow M$ is a trivial bundle. An **orientation** is a choice of trivializing section, and a manifold together with an orientation is said to be **oriented**. Note that, since $\Lambda^k(E^*) = (\Lambda^k E)^*$, E is orientable if and only if E^* is orientable.

An m-manifold M is **orientable** if and only if its tangent bundle is orientable. Equivalently, a manifold is orientable when its cotangent bundle is orientable, so when there is a nowhere vanishing m-form. A choice of nowhere vanishing m-form is called an **orientation** of M or also a **volume form** on M. A coordinate chart $\phi : \mathcal{U} \longrightarrow \mathcal{V}$ is oriented if it pulls-back the orientation of M to the canonical orientation of $\mathbb{R}^m$. A manifold is orientable precisely when it has an atlas of oriented coordinate charts.

If M is oriented, there is a unique linear map the **integral**

$$\int : \mathcal{C}^\infty_c(M; \Lambda^m M) \longrightarrow \mathbb{R}$$

which is invariant under diffeomorphims and coincides in local coordinates with the Lebesgue integral. That is, if $\omega \in \Omega^m M$ is supported in a oriented coordinate chart $\phi : \mathcal{U} \longrightarrow \mathcal{V}$ then $\phi^*\omega \in \Omega^m(\mathbb{R}^m)$ and so we can take its Lebesgue integral. The assertion is that the result is independent of the choice of oriented coordinate chart, and the proof follows from (1.6) and the usual change of variable formula for the Lebesgue integral. By using a partition of unity, integration of any m-form reduces to integration of m-forms supported in oriented coordinate charts. It is convenient to formally extend the integral to a functional on forms of all degrees, by setting it equal to zero on all forms of less than maximal degree.

What happens when a manifold is not orientable? At each point $p \in M$, the set of equivalence classes of orientations of TM, $\mathscr{O}_p$, has two elements, say $o_1(p)$ and

$o_2(p)$. We define the **orientable double cover of** M to be the set

$$\mathscr{O}r(M) = \{(p, o_i(p)) : p \in M, o_i(p) \in \mathscr{O}_p\}.$$

The manifold structure on M easily extends to show that $\widetilde{M}$ is a smooth manifold. Indeed, given a coordinate chart $\phi : \mathcal{U} \longrightarrow \mathcal{V}$ of M we set, for each $q \in \mathcal{U}$,

$$\widetilde{\phi}(q) = (\phi(q), [\phi^{-1}(e_1 \wedge \cdots \wedge e_m)]) \in \mathscr{O}r(M)$$

and we get a coordinate chart on $\widetilde{M}$. The mapping $\pi : \widetilde{M} \longrightarrow M$ that sends $(p, o_i(p))$ to p is smooth and surjective. Moreover if $\mathcal{V}$ is a coordinate neighborhood in M then $\pi^{-1}(\mathcal{V}) = \mathcal{V}_1 \cup \mathcal{V}_2$ is the union of two disjoint open sets in $\mathscr{O}r(M)$ and π restricted to each $\mathcal{V}_i$ is a diffeomorphism onto $\mathcal{V}$, hence the term 'double cover'. It is easy to see that $\mathscr{O}r(M)$ is always orientable (at each point $(p, o_i(p))$ choose the orientation corresponding to $o_i(p)$). If M is connected and not orientable then $\mathscr{O}r(M)$ is connected and orientable, and we can often pass to $\mathscr{O}r(M)$ if we need an orientation. If M is orientable, then the same construction of $\mathscr{O}r(M)$ yields a disconnected manifold consisting of two copies of M.

Orientations are one place where the theory of complex vector bundles differs from that of general real vector bundles. Complex vector bundles are always orientable.

Indeed, first note that a $\mathbb{C}$-vector space W of complex dimension k has a canonical orientation as a $\mathbb{R}$-vector space of real dimension $2k$. We just take any basis of W over $\mathbb{C}$, say $b = \{v_1, \ldots, v_k\}$, and then take $\widehat{b} = \{v_1, iv_1, v_2, iv_2, \ldots, v_k, iv_k\}$ as a basis for W over $\mathbb{R}$. The claim is that if $b' = \{w_1, \ldots, w_k\}$ is any other basis of W over $\mathbb{C}$, then $\widehat{b'} = \{w_1, iw_1, \ldots, w_k, iw_k\}$ gives us the same orientation for W as a $\mathbb{R}$-vector space, i.e., the change of basis matrix $\widehat{B}$ from $\widehat{b}$ to $\widehat{b'}$ has positive determinant. The proof is just to note that, if B is the complex change of basis matrix from b to b' then

$$\det \widehat{B} = (\det B)(\overline{\det B}) > 0.$$

As the orientation is canonical, every $\mathbb{C}^k$-vector bundle is orientable as a $\mathbb{R}^{2k}$-vector bundle.

1.11 Exterior derivative

If f is a smooth function on M, its **exterior derivative** is a 1-form $df \in \mathcal{C}^\infty(M; T^*M)$ defined by

$$df(V) = V(f), \qquad \text{for all } V \in \mathcal{C}^\infty(M; TM).$$

If $\omega \in \mathcal{C}^\infty(M; T^*M)$, its exterior derivative is a 2-form $d\omega \in \mathcal{C}^\infty(M; \Lambda^2 T^*M)$ defined by

$$d\omega(V, W) = V(\omega(W)) - W(\omega(V)) - \omega([V, W]), \text{ for all } V, W \in \mathcal{C}^\infty(M; TM).$$

In the same way, if $\omega \in \Omega^k(M)$, we define its exterior derivative to be the $(k+1)$-form $d\omega \in \Omega^{k+1}(M)$ defined by

$$\begin{aligned} d\omega(V_0, \ldots, V_k) &= \sum_{i=0}^{p} (-1)^i V_i(\omega(V_0, \ldots, \widehat{V_i}, \ldots, V_p)) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([V_i, V_j], V_0, \ldots, \widehat{V_i}, \ldots, \widehat{V_j}, \ldots, V_p) \end{aligned}$$

The exterior derivative thus defines a graded map

$$d : \Omega^*(M) \longrightarrow \Omega^*(M).$$

It has the property that $d^2 = 0$. It also distributes over the wedge product respecting the grading, in that

$$\omega \in \Omega^k(M), \eta \in \Omega^\ell(M) \implies d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge d\eta. \tag{1.7}$$

The exterior derivative also commutes with pull-back along a smooth map $F : M_1 \longrightarrow M_2$. Indeed, note that for a function on M_2, $f \in \mathcal{C}^\infty(M_2)$, this follows from the chain rule

$$d(F^*f)(V) = d(f \circ F)(V) = D(f \circ F)(V) = Df(DF(V)) = df(DF(V)) = F^*(df)(V)$$

and, using this, it follows for forms of arbitrary degree from (1.5) and (1.7).

A k-form ω is called **closed** if $d\omega = 0$ and **exact** if there is a $(k-1)$-form η such that $d\eta = \omega$. The space of k-forms is an infinite dimensional vector space over $\mathbb{R}$, as are the spaces of closed and exact forms. However we will show that, for a compact manifold M and any k, the quotient vector space

$$\mathrm{H}^k_{\mathrm{dR}}(M) = \frac{\text{closed } k\text{-forms}}{\text{exact } k\text{-forms}} = \frac{\ker\big(d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)\big)}{\operatorname{im}\big(d : \Omega^{k-1}(M) \longrightarrow \Omega^k(M)\big)}$$

is finite dimensional. This $\mathbb{R}$-vector space is known as the **k-th de Rham cohomology group of** M. These groups are one of the main objects of study in this course. We will come back to them after we build up more tools.

If M is an oriented manifold with boundary, Stokes' theorem is the fact that

$$\int_M d\omega = \int_{\partial M} \omega, \text{ for all } \omega \in \Omega^{m-1}_c(M).$$

This compact statement generalizes the fundamental theorem of calculus, Green's theorem, and Gauss' theorem. If M does not have boundary, then Stokes' theorem says that

$$\int_M d\omega = 0, \text{ for all } \omega \in \Omega_c^{m-1}(M).$$

1.12 Differential operators

It is clear what we mean by a differential operator of order n on $\mathbb{R}^m$, namely a polynomial of degree n in the variables $\partial_{y_1}, \partial_{y_2}, \dots, \partial_{y_m}$,

$$\sum_{\substack{j_1, j_2, \dots, j_m \\ j_1 + \dots + j_m = n}} a_{j_1, \dots, j_m}(y_1, \dots, y_m) \partial_{y_1}^{j_1} \partial_{y_2}^{j_2} \cdots \partial_{y_m}^{j_m}. \tag{1.8}$$

(The degree of this operator is the largest value of $j_1 + j_2 + \ldots + j_m$ for which $a_{j_1,\dots,j_m}$ is not zero, which we are assuming is n.) This operator is smooth when the coefficient functions $a_{j_1,\dots,j_m}(y_1, \dots, y_m)$ are all smooth. The set of all differential operators of order at most n, is denoted $\mathrm{Diff}^n(\mathbb{R}^m)$.

This is a good moment to introduce multi-index notation. Let us agree, for all vectors $\alpha = (j_1, \dots, j_m)$, on the equivalence of the symbols

$$\partial_y^\alpha = \partial_{y_1}^{j_1} \partial_{y_2}^{j_2} \cdots \partial_{y_m}^{j_m},$$

and let us use $|\alpha|$ to denote $j_1 + \ldots + j_m$. Then we can write the expression above more succinctly as

$$P \in \mathrm{Diff}^n(\mathbb{R}^m) \iff P = \sum_{|\alpha| \leq n} a_\alpha(y) \partial_y^\alpha.$$

On a manifold, a differential operator of order at most n is a linear map

$$P : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)$$

that in any choice of local coordinates has the form (1.8). For instance, a vector field V induces a linear map (a derivation) on smooth functions

$$V : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)$$

and this is exactly what we mean by a first order differential operator on M. In fact, a better description of differential operators of order at most n on M is to take the span of powers of $\mathcal{C}^\infty(M; TM)$,

$$\mathrm{Diff}^n(M) = \operatorname*{span}_{0 \leq j \leq n} (\mathcal{C}^\infty(M; TM))^j$$

where we agree that $\mathcal{C}^\infty(M;TM)^0 = \mathcal{C}^\infty(M)$. Notice that the fact that the degree of a differential operator is well-defined comes from the fact that the commutator of two vector fields is again a vector field.

The exterior derivative is also a first order differential operator, but now between sections of vector bundles over M, e.g., for each k,

$$d : \mathcal{C}^\infty(M;\Lambda^k T^*M) \longrightarrow \mathcal{C}^\infty(M;\Lambda^{k+1}T^*M).$$

In general, a differential operator of order at most n acting between sections of two vector bundles E and F is a linear map

$$\mathcal{C}^\infty(M;E) \longrightarrow \mathcal{C}^\infty(M;F)$$

that in any local coordinates that trivialize both E and F has the form (1.8) where we now require the functions $a_\alpha(y)$ to be sections of $\mathrm{Hom}(E,F)$.

There is a very nice characterization (or definition if you prefer) of differential operators due to Peetre[3]. First, we say that a linear operator

$$T : \mathcal{C}^\infty(M;E) \longrightarrow \mathcal{C}^\infty(M;F)$$

is a **local operator** if whenever $s \in \mathcal{C}^\infty(M;E)$ vanishes in an open set $\mathcal{U} \subseteq M$, Ts also vanishes in that open set,

$$s\big|_{\mathcal{U}} = 0 \implies (Ts)\big|_{\mathcal{U}} = 0.$$

Peetre showed that all linear local operators are differential operators.

1.13 Exercises

Exercise 1.
Show that whenever M_1 and M_2 are smooth manifolds, $M_1 \times M_2$ is too. Prove that $T(M_1 \times M_2) = TM_1 \times TM_2$.

Exercise 2.
Instead of defining the cotangent bundle in terms of the tangent bundle, we can do things the other way around. At each point p of a manifold M, let

$$\mathcal{I}_p = \{u \in \mathcal{C}^\infty(M) : u(p) = 0\}$$

[3] Peetre, Jaak *Une caractérisation abstraite des opérateurs différentiels.* Math. Scand. **7** 1959 211-218.

be the ideal of functions vanishing at p. Let $\mathcal{I}_p^2$ be the linear span of products of pairs of elements in $\mathcal{I}_p$, and define

$$\mathcal{T}_p^* M = \mathcal{I}_p/\mathcal{I}_p^2, \quad \mathcal{T}^* M = \bigcup_{p\in M} \mathcal{T}_p^* M.$$

Show that, together with the obvious projection map, this defines a vector bundle isomorphic to the cotangent bundle.

Exercise 3.
Show that a $\mathbb{R}^k$-bundle $E \xrightarrow{\pi} M$ is trivial if and only if there are k sections $s_1, \dots, s_k$ such that, at each $p \in M$,

$$\{s_1(p), \dots, s_k(p)\} \text{ is linearly independent.}$$

Exercise 4.
If $E \xrightarrow{\pi_E} M$ is a subbundle of $F \xrightarrow{\pi_F} M$ show that there is a bundle $F/E \longrightarrow M$ whose fiber over $p \in M$ is $\pi_F^{-1}(p)/\pi_E^{-1}(p)$

Exercise 5.
Given vector bundles $E \longrightarrow M$ and $F \longrightarrow M$, assume that $f \in \mathcal{C}^\infty(M; \mathrm{Hom}(E, F))$ is such that

$$\dim\ker f(p) \text{ is independent of } p \in M.$$

Prove that the kernels of f form a vector bundle over M (a subbundle of E) and the cokernels of f also form a vector bundle over M (a quotient of F.)

Exercise 6.
(*Poincaré's Lemma for 1-forms.*)
Let

$$\omega = a(x, y, z)\ dx + b(x, y, z)\ dy + c(x, y, z)\ dz$$

be a smooth 1-form in $\mathbb{R}^3$ such that $d\omega = 0$. Define $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ by

$$f(x, y, z) = \int_0^1 (a(tx, ty, tz)x + b(tx, ty, tz)y + c(tx, ty, tz)z)\ dt.$$

Show that $df = \omega$.
(Thus on $\mathbb{R}^3$ (also $\mathbb{R}^n$) a 1-form ω satisfies $d\omega = 0 \iff \omega = df$ for some f. It turns out that this is a reflection of the 'contractibility' of $\mathbb{R}^n$.)
Hint: First use $d\omega = 0$ to show that

$$\frac{\partial b}{\partial x} = \frac{\partial a}{\partial y}, \quad \frac{\partial c}{\partial x} = \frac{\partial a}{\partial z}, \quad \frac{\partial b}{\partial z} = \frac{\partial c}{\partial y}$$

and then use the identity

$$\begin{aligned} a(x, y, z) &= \int_0^1 \frac{\partial}{\partial t}(a(tx, ty, tz)t)\ dt \\ &= \int_0^1 a(tx, ty, tz)\ dt + \int_0^1 t\left(x\frac{\partial a}{\partial x} + y\frac{\partial a}{\partial y} + z\frac{\partial a}{\partial z}\right)\ dt. \end{aligned}$$

Exercise 7.
Show that any smooth map $F : M_1 \longrightarrow M_2$ induces, for any $k \in \mathbb{N}_0$, a map in cohomology

$$F^* : \mathrm{H}^k_{\mathrm{dR}}(M_2) \longrightarrow \mathrm{H}^k_{\mathrm{dR}}(M_1)$$

Exercise 8.
Show that the wedge product and the integral descend to cohomology. That is, show that the value of $\int \omega$ and the cohomology class of $\omega \wedge \eta$ only depends on the cohomology classes of ω and η.

Exercise 9.
Show that a linear map $T : \mathcal{C}^\infty(M; E) \longrightarrow \mathcal{C}^\infty(M; F)$ corresponds to a section of $\mathcal{C}^\infty(M; \mathrm{Hom}(E, F))$ if and only if T is linear over $\mathcal{C}^\infty(M)$.

1.14 Bibliography

There are many wonderful sources for differential geometry. The first volume of Spivak, *A comprehensive introduction to differential geometry* is written in a conversational style that makes it a pleasure to read. The other four volumes are also very good. I also recommend Lee, *Introduction to smooth manifolds.*

For vector bundles, the books Milnor and Stasheff, *Characteristic classes* and Atiyah, *K-theory* are written by fantastic expositors, as is the book Bott and Tu, *Differential forms in algebraic topology.*

Lecture 2

A brief introduction to Riemannian geometry

2.1 Riemannian metrics

A **metric** on a vector bundle $E \longrightarrow M$ is a smooth family of inner products, one on each fiber of E. We make sense of smoothness by using the bundle $S^2(E) \longrightarrow M$ of symmetric bilinear forms on the fibers of E. Formally, a metric g_E on E is a section of the vector bundle $S^2(E) \longrightarrow M$ that is non-degenerate and positive definite. In particular we have an $\mathcal{C}^\infty(M)$-valued inner product on the sections of E, so a smooth map

$$g_E : \mathcal{C}^\infty(M; E) \times \mathcal{C}^\infty(M; E) \longrightarrow \mathcal{C}^\infty(M)$$

that is linear in each entry, symmetric, and satisfies

$$g_E(s, s) \geq 0 \text{ and } g_E(s, s) = 0 \iff s = 0.$$

A metric g on the tangent bundle of M is called a **Riemannian metric,** and the pair (M, g) is called a **Riemannian manifold.**

Any vector bundle admits a metric. Indeed, on a local trivialization we can just put the Euclidean inner product. Covering the manifold with local trivializations, we can use a partition of unity to put these inner products together into a bundle metric.

Given a metric g_E on E we can use the inner product $g_E(p)$ to identify a vector $v \in E_p$ with a vector in E^* by

$$(\widetilde{g}_E(p))(v)(w) = g_E(p)(v, w) \text{ for all } w \in E_p. \tag{2.1}$$

This extends to a bundle isomorphism

$$\widetilde{g}_E : E \longrightarrow E^*.$$

Conversely, any bundle isomorphism between E and E^* defines by (2.1) an inner product on E (and on E^*), though not necessarily positive definite.

In particular a Riemannian metric g on M induces a bundle isomorphism between TM and T^*M, known as the 'musical isomorphism'. A vector field V is associated a differential 1-form via g, denoted $V^\flat$, and a 1-form ω is associated a vector field via g, denoted $\omega^\sharp$.

Incidentally, this is how the gradient is defined on a Riemannian manifold. Given a function $f \in \mathcal{C}^\infty(M)$, we get a one form $df \in \mathcal{C}^\infty(M; TM)$, and we use the metric to turn this into a vector field

$$\nabla f = (df)^\sharp \in \mathcal{C}^\infty(M; TM).$$

Thus, the gradient satisfies (and is defined by)

$$g(\nabla f, V) = df(V) = V(f), \text{ for all } V \in \mathcal{C}^\infty(M; TM).$$

Let us look at the objects in local coordinates. Let $\mathcal{V}$ be a local coordinate chart of M trivializing E, so that we have coordinates

$$\{y_1, \dots, y_m, s_1, \dots, s_k\} \tag{2.2}$$

where $y_1, \dots, y_m$ are local coordinates on M, and $s_1, \dots, s_k$ are linearly independent sections of E depending on y. A metric is a smooth family of $k \times k$ symmetric positive definite matrices

$$y \mapsto g_E(y) = ((g_E)_{ab}(y)), \text{ where } (g_E)_{ab} = g_E(s_a, s_b).$$

Any section of E, $\sigma : \mathcal{V} \longrightarrow E$, can be written

$$\sigma(y) = \sum_{a=1}^{k} \sigma^a(y) s_a$$

for some smooth functions σ^a. It is common to write this more succinctly using the '**Einstein summation convention**' as

$$\sigma(y) = \sigma^a(y) s_a$$

with the understanding that *a repeated index that occurs as both a subindex and a superindex should be summed over.* If $\sigma = \sigma^a s_a$ and $\tau = \tau^a s_a$ are two sections of E, their inner product with respect to g_E is

$$g_E(\sigma, \tau) = \sigma^a \tau^b (g_E)_{ab}.$$

It is very convenient, with regards to the Einstein convention, to denote the entries of the inverse matrix of $(g_E)_{ab}$ by $(g_E)^{ab}$, thus for instance we have

$$(g_E)^{ab}(g_E)_{bc} = \delta^a_c.$$

The metric g_E determines a metric g_{E^*} on E^* by the relation

$$g_{E^*}(\widetilde{g}_E(\sigma), \widetilde{g}_E(\tau)) = g_E(\sigma, \tau), \tag{2.3}$$

and in a local trivialization this corresponds to taking the inverse matrix of $(g_E)_{ab}$. Indeed, in these same coordinates (2.2) we have a natural local trivialization of E^*, namely

$$\{y_1, \ldots, y_m, s^1, \ldots, s^k\}, \text{ where } s^j \in \mathcal{C}^\infty(\mathcal{V}; E^*) \text{ satisfy } s^i s_j = \delta^i_j.$$

Then the map $\widetilde{g}_E$ satisfies

$$(\widetilde{g}_E)(s_a)(s_b) = g_E(s_a, s_b) = g_{ab}, \text{ so } (\widetilde{g}_E)(s_a) = (g_E)_{ab} s^b$$

and hence $s^b = (g_E)^{ab}\widetilde{g}_E(s_a)$. Thus we have

$$\begin{aligned}(g_{E^*})_{ab} = g_{E^*}(s^a, s^b) &= g_{E^*}((g_E)^{ac}\widetilde{g}_E(s_c), (g_E)^{bn}\widetilde{g}_E(s_n)) \\ &= (g_E)^{ac}(g_E)^{bn}(g_E)_{cn} = (g_E)^{ac}\delta^b_c = (g_E)^{ab}\end{aligned}$$

as required.

Given a smooth map $F : M_1 \longrightarrow M_2$ we can pull-back a metric g on M_2 to define a section of $S^2(TM_1) \longrightarrow M_1$,

$$(F^*g)(V_1, V_2) = g(DF(V_1), DF(V_2)).$$

If F is an immersion, then F^*g is a metric on M_1. In particular, submanifolds of a Riemannian manifold inherit metrics.

The simplest example of a Riemannian m-manifold is $\mathbb{R}^m$ with the usual Euclidean inner product on the fibers of

$$T\mathbb{R}^m = \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}^m.$$

From the previous paragraph it follows that all submanifolds of $\mathbb{R}^m$ inherit a metric from the one on $\mathbb{R}^m$. A difficult theorem of Nash, the Nash embedding theorem, shows that all Riemannian manifolds can be embedded into $\mathbb{R}^N$ for some N in such a way that the metric coincides with the one inherited from $\mathbb{R}^N$. So we can think of an arbitrary Riemannian manifold as a submanifold of a Euclidean space.

If (M_1, g_1) and (M_2, g_2) are Riemannian manifolds, a map $F : M_1 \longrightarrow M_2$ is a **local isometry** if it is a local diffeomorphism and $F^*g_2 = g_1$. An **isometry** between two Riemannian manifolds is a bijective local isometry. Riemannian geometry is the study of invariants of isometry classes of manifolds.

A metric on $E \longrightarrow M$ allows us to define the length of a vector $v \in E_p$ by

$$|v|_{g_E} = \sqrt{g_E(p)(v, v)}$$

and the angle θ between two vectors $v, w \in E_p$ satisfies

$$g_E(p)(v, w) = |v|_{g_E} |w|_{g_E} \cos\theta.$$

Given a curve $\gamma : [a, b] \longrightarrow M$ on M, and a Riemannian metric g, the length of γ is defined to be

$$\ell(\gamma) = \int_a^b |\gamma'(t)|_g \, dt.$$

This is independent of the chosen parametrization of γ. We can use the length of curves to define the distance between two points, $p, q \in M$, to be

$$d(p, q) = \inf\{\ell(\gamma) : \gamma \in \mathcal{C}^\infty([0, 1], M), \gamma(0) = p, \gamma(1) = q\} \tag{2.4}$$

it is easy to see that

$$d(p, q) \geq 0, \quad d(p, q) = d(q, p), \quad d(p, q) \leq d(p, r) + d(r, q)$$

and we will show below that $d(p, q) = 0 \implies p = q$, so that (M, d) is a metric space.

2.2 Parallel translation

On $\mathbb{R}^m$, and more generally on a parallelizable manifold, it is clear what we mean by a constant vector field. Given a tangent vector at a point in one of these spaces, we can 'parallel translate' it to the whole space just taking the constant vector field.

To try to make sense of this on a manifold, suppose we have a smooth curve $\gamma : [0,1] \longrightarrow M$ and a vector field $V \in \mathcal{C}^\infty(M;TM)$, we would like to say that V is constant (or parallel) along γ if the map

$$V \circ \gamma : [0,1] \longrightarrow TM$$

is constant. The problem with making sense of this is that TM is not a vector space but a collection of vector spaces, so $V(\gamma(t)) \in T_{\gamma(t)}M$ belongs to a different vector space for each t. The same problem occurs with trying to make sense of a constant section of a vector bundle $E \longrightarrow M$. The solution is to use a **connection**.

A connection on a vector bundle $E \longrightarrow M$ is a map

$$\nabla^E : \mathcal{C}^\infty(M;TM) \times \mathcal{C}^\infty(M;E) \longrightarrow \mathcal{C}^\infty(M;E)$$

which is denoted $\nabla^E(V,s) = \nabla^E_V s$, is $\mathbb{R}$-linear in both variables, i.e., for all $a_i \in \mathbb{R}$, $V_i \in \mathcal{C}^\infty(M;TM)$, $s_i \in \mathcal{C}^\infty(M;E)$,

$$\nabla^E_{a_1V_1+a_2V_2} s = a_1\nabla^E_{V_1} s + a_2\nabla^E_{V_2} s, \quad \nabla^E_V(a_1s_1 + a_2s_2) = a_1\nabla^E_V s_1 + a_2\nabla^E_V s_2$$

and satisfies, for $f \in \mathcal{C}^\infty(M)$,

$$\nabla^E_{fV} s = f\nabla^E_V s, \quad \nabla^E_V(fs) = V(f)s + f\nabla^E_V s. \tag{2.5}$$

It follows that the value of $\nabla^E_V s$ at a point $p \in M$ depends only on the value of V at p and the values of s in a neighborhood of p. We refer to $\nabla^E_V s$ as the **covariant derivative of s in the direction of V**.

On a trivial bundle $\mathbb{R}^k \times M \longrightarrow M$, a vector field is a map $M \longrightarrow \mathbb{R}^k$ and the usual derivative on $\mathbb{R}^k$ defines a covariant derivative on $\underline{\mathbb{R}^k}$.

In a local trivialization of E,

$$\{y_1, \ldots, y_m, s_1, \ldots, s_k\}$$

the connection is determined by mk^2 functions,

$$\Gamma^b_{ia}, \text{ where } 1 \leq i \leq m, \quad 1 \leq a, b \leq k$$

known as the **Christoffel symbols**, defined by

$$\nabla^E_{\partial_{y_i}} s_a = \Gamma^b_{ia} s_b$$

where we are using the Einstein summation convention as above. Once we know the functions Γ^b_{ia} we can find $\nabla_V s$ at any point in this local trivialization by using (2.5).

We can answer our original question by saying that, for a given curve $\gamma : [a,b] \longrightarrow M$ a section of E, $s : M \longrightarrow E$, is **parallel along** γ (with respect to ∇^E) if

$$\nabla^E_{\gamma'(t)} s = 0$$

at all points $\gamma(t)$. Let us see write this equation in local coordinates as above (2.2). We can write

$$s(t) := s(\gamma(t)) = \sigma^a(t) s_a, \quad \gamma'(t) = (\gamma^i)'(t)\partial_{y_i}$$

and then

$$\nabla^E_{\gamma'(t)} s = (\partial_t \sigma^a) s_a + ((\gamma^i)'(t))\sigma^a \Gamma^b_{ia} s_b$$

grouping together the coefficients of s_a, we see that this will vanish precisely when

$$\partial_t \sigma^a + ((\gamma^i)'(t))\sigma^c \Gamma^a_{ic} = 0 \text{ for all } a. \tag{2.6}$$

This local expression shows that it is always possible to extend a vector in E_p to a covariantly constant section of E along any curve through p. Indeed, given any such curve $\gamma : [a,b] \longrightarrow M$, with $p = \gamma(t_0)$ and any vector $s_0 \in E_{\gamma(t_0)}$, we can find s by solving (2.6) for its coefficients $\sigma^1, \ldots, \sigma^k$. As this is a linear system of ODE's, there is a unique solution defined on all of $[a,b]$. We call this solution **the parallel transport of s_0 along γ.**

Parallel transport along γ defines a linear isomorphism between fibers of $E \longrightarrow M$,

$$\mathcal{T}_\gamma : E_a \longrightarrow E_b.$$

This is particularly interesting when γ is a loop at a point $p \in M$. The possible linear automorphisms of E_p resulting from parallel transport along loops at p form a group, the **holonomy group of ∇**. Different points in M yield isomorphic groups so, as long as M is connected, this group is independent of the base point p.

Remark 3. We can use parallel transport to show that pull-back bundles under homotopic maps are isomorphic. Indeed, assume that $f : M_1 \longrightarrow M_2$ and $g : M_1 \longrightarrow M_2$ are homotopic smooth maps, so that there is a smooth map

$$H : [0,1] \times M_1 \longrightarrow M_2 \text{ s.t. } H(0,\cdot) = f(\cdot) \text{ and } H(1,\cdot) = g(\cdot)$$

and let $E \longrightarrow M_2$ be a vector bundle. We want to show that $f^*E \cong g^*E$, and by considering H^*E we see that we can reduce to the case $M_2 = [0,1] \times M_1$, with

$$f = i_0 : M_1 \longrightarrow [0,1] \times M_1, \quad i_0(p) = (0,p)$$
$$g = i_1 : M_1 \longrightarrow [0,1] \times M_1, \quad i_1(p) = (1,p).$$

So let $F \longrightarrow [0,1] \times M_1$ be a vector bundle. Choose a connection on F and for each $p \in M_1$ let $\mathcal{T}_p$ be parallel transport along the curve $t \mapsto (t,p)$, then the map

$$\Phi : i_0^* F \longrightarrow i_1^* F, \quad \Phi(p,v) = (p, \mathcal{T}_p(v))$$

is a vector bundle isomorphism.

Next assume that we have both a metric g_E and a connection ∇^E on E. We will say that these objects are compatible if, for any sections σ and τ of E, and any vector field $V \in \mathcal{C}^\infty(M;TM)$,

$$V(g_E(\sigma,\tau)) = g_E(\nabla^E_V \sigma, \tau) + g_E(\sigma, \nabla^E_V \tau). \tag{2.7}$$

We also say in this case that ∇^E is a **metric connection.**

Parallel transport with respect to a metric connection is an isometry. That is, if $\gamma : [a,b] \longrightarrow M$ is a curve on M and σ and τ are parallel along γ, then

$$\partial_t g_E(\sigma(t), \tau(t)) = 0.$$

In fact this follows immediately from the metric condition since the derivative with respect to t is precisely the derivative with respect to the vector field $\gamma'(t)$. In local coordinates, the connection is metric if and only if the Christoffel symbols satisfy

$$\partial_{y_i}(g_E)_{ab} = \Gamma^c_{ia}(g_E)_{cb} + \Gamma^c_{ib}(g_E)_{ac}.$$

Given the matrix-valued function g_E, we can always find functions Γ^b_{ia} that satisfy these relations. In other words given a metric on a trivial bundle, there is always a metric connection. Using a partition of unity it follows that for any metric on a bundle E there is a metric connection.

A connection on E is, for each vector field $V \in \mathcal{C}^\infty(M;TM)$, a first-order differential operator on sections of E,

$$\nabla^E_V : \mathcal{C}^\infty(M;E) \longrightarrow \mathcal{C}^\infty(M;E).$$

If ∇^E and $\widetilde{\nabla}^E$ are two connections on E, then their difference is linear over $\mathcal{C}^\infty(M)$,

$$\nabla^E_V - \widetilde{\nabla}^E_V : \mathcal{C}^\infty(M;E) \longrightarrow \mathcal{C}^\infty(M;E) \text{ satisfies}$$
$$(\nabla^E_V - \widetilde{\nabla}^E_V)(fs) = f(\nabla^E_V - \widetilde{\nabla}^E_V)(s) \text{ for all } f \in \mathcal{C}^\infty(M), s \in \mathcal{C}^\infty(M;E)$$

and hence a zero-th order differential operator or equivalently, a section of the homomorphism bundle of E,

$$\nabla^E_V - \widetilde{\nabla}^E_V \in \mathcal{C}^\infty(M; \mathrm{Hom}(E)).$$

We can also incorporate the dependence on the vector field V and write

$$\nabla^E - \widetilde{\nabla}^E \in \mathcal{C}^\infty(M; T^*M \otimes \mathrm{Hom}(E)).$$

Conversely, it is easy to see that whenever ∇^E is a connection on E and $\Phi \in \mathcal{C}^\infty(M; T^*M \otimes \mathrm{Hom}(E))$ is a section of $T^*M \otimes \mathrm{Hom}(E)$, the sum $\nabla^E + \Phi$ is again a connection on E. We say that the space of connections on E is an *affine space modeled on* $\mathcal{C}^\infty(M; T^*M \otimes \mathrm{Hom}(E))$. In particular, any bundle has infinitely many connections on it.

2.3 Levi-Civita connection

Among the infinitely many connections on the tangent bundle of a Riemannian manifold, there is a unique connection that is compatible with the metric and satisfies an additional symmetry property.

Theorem 1 (Fundamental theorem of Riemannian geometry)**.** *Let (M, g) be a Riemannian manifold, there is a unique metric connection ∇ on TM satisfying*

$$\nabla_V W - \nabla_W V - [V, W] = 0 \textit{ for all } V, W \in \mathcal{C}^\infty(M; TM). \tag{2.8}$$

We say that a connection (necessarily on TM) that satisfies on (2.8) is **symmetric** or 'torsion-free'. The unique metric torsion-free connection of a metric is called the **Levi-Civita connection.**

Proof. Given any three vector fields $U, V, W \in \mathcal{C}^\infty(M; TM)$, we can apply the metric property (2.7) to

$$Ug(V, W) + Vg(W, U) - Wg(U, V)$$

and then use the torsion-free property to cancel out all but one covariant derivative to arrive at the **Koszul formula**:

$$g(\nabla_U V, W) = \frac{1}{2}\Big(Ug(V, W) + Vg(W, U) - Wg(U, V) \\ - g(U, [V, W]) + g(V, [W, U]) + g(W, [U, V])\Big),$$

which proves uniqueness. It is easy to check that this formula defines a symmetric metric connection, establishing existence. □

In local coordinates, $\{y_1, \ldots, y_m, \partial_{y_1}, \ldots, \partial_{y_k}\}$, a connection on TM is symmetric precisely when its Christoffel symbols satisfy

$$\Gamma_{ij}^k = \Gamma_{ji}^k.$$

Note that the coordinate vector fields ∂_{y_i} satisfy Schwarz' theorem, $[\partial_{y_i}, \partial_{y_j}] = 0$, so from Koszul's formula the Christoffel symbols of the Levi-Civita connection are given by

$$\Gamma^k_{ij} = \frac{1}{2} g^{k\ell} \left(\partial_i g_{\ell j} + \partial_j g_{i\ell} - \partial_\ell g_{ij} \right).$$

2.4 Associated bundles

We saw above (2.3) that a metric on E determines a dual metric on E^* by requiring that the bundle isomorphism

$$\widetilde{g}_E : E \longrightarrow E^*$$

be an isometry. A connection on E also naturally induces a connection on E^*. Given sections s of E and η of E^*, we can 'contract' or pair them to get a function $\eta(s)$ on M, and we demand that the putative connection on E^* satisfy the corresponding Leibnitz rule,

$$V(\eta(s)) = (\nabla^{E^*}_V \eta)(s) + \eta(\nabla^E_V s).$$

Thus we define $\nabla^{E^*}_V$ by

$$(\nabla^{E^*}_V \eta)(s) = V(\eta(s)) - \eta(\nabla^E_V s).$$

If $E \longrightarrow M$ and $F \longrightarrow M$ are vector bundles over M endowed with metrics g_E and g_F then we have induced metrics

$$g_{E\oplus F} \text{ on } E \oplus F \longrightarrow M, \quad g_{E\otimes F} \text{ on } E \otimes F \longrightarrow M$$

determined by

$$g_{E\oplus F}(\sigma \oplus \alpha, \tau \oplus \beta) = g_E(\sigma, \tau) + g_F(\alpha, \beta),$$
$$g_{E\otimes F}(\sigma \otimes \alpha, \tau \otimes \beta) = g_E(\sigma, \tau) g_F(\alpha, \beta),$$

for all $\sigma, \tau \in \mathcal{C}^\infty(M; E)$, $\alpha, \beta \in \mathcal{C}^\infty(M; F)$. Similarly, given connections ∇^E and ∇^F, we have induced connections

$$\nabla^{E\oplus F} \text{ on } E \oplus F \longrightarrow M, \quad \nabla^{E\otimes F} \text{ on } E \otimes F \longrightarrow M.$$

determined by the corresponding Leibnitz rules,

$$\nabla^{E\oplus F}_V (\sigma \oplus \alpha) = \nabla^E_V \sigma \oplus \nabla^F_V \alpha$$
$$\nabla^{E\otimes F}_V (\sigma \otimes \alpha) = \nabla^E_V \sigma \otimes \alpha + \sigma \otimes \nabla^F_V \alpha,$$

for all $V \in \mathcal{C}^\infty(M; TM)$, $\sigma \in \mathcal{C}^\infty(M; E)$, and $\alpha \in \mathcal{C}^\infty(M; F)$.

In particular, connections on E and F induce a connection on $\mathrm{Hom}(E,F) = E^* \otimes F \longrightarrow M$, satisfying

$$(\nabla_V^{\mathrm{Hom}(E,F)}\Phi)(\sigma) = \nabla_V^F(\Phi(\sigma)) - \Phi(\nabla_V^E\sigma)$$

for all $V \in \mathcal{C}^\infty(M;TM)$, $\Phi \in \mathcal{C}^\infty(M;\mathrm{Hom}(E,F))$, and $\sigma \in \mathcal{C}^\infty(M;E)$.

Using these induced connections, it makes sense to take the covariant derivative of a metric on E as a section of $E^* \otimes E^*$,

$$(\nabla_V^{E^*\otimes E^*} g_E)(\sigma,\tau) = V(g_E(\sigma,\tau)) - g_E(\nabla_V^E\sigma,\tau) - g_E(\sigma,\nabla_V^E\tau).$$

Thus we find that a connection is compatible with a metric if and only if the metric is covariantly constant with respect to the induced connection.

We also have induced metrics and connections on pull-back bundles. Let $f : M_1 \longrightarrow M_2$ be a smooth map and $E \longrightarrow M_2$ a vector bundle with metric g_E and connection ∇^E. Any section of E, $\sigma : M_2 \longrightarrow E$, can be pulled-back to a section of f^*E,

$$(f^*\sigma)(p) = (p, \sigma(f(p))) \in (f^*E)_p, \text{ for all } p \in M_1.$$

Moreover these pulled-back sections generate $\mathcal{C}^\infty(M_1; f^*E)$ as a $\mathcal{C}^\infty(M_1)$-module in that

$$\mathcal{C}^\infty(M_1; f^*E) = \mathcal{C}^\infty(M_1) \cdot f^*\mathcal{C}^\infty(M_2; E).$$

Thus, to specify a metric and connection on f^*E, it suffices to describe their actions on products of functions on M_1 and pulled-back sections. Thus for any $h_1, h_2 \in \mathcal{C}^\infty(M_1)$, and $\sigma_1, \sigma_2 \in \mathcal{C}^\infty(M_2; E)$ we define

$$g_{f^*E}(h_1 f^*\sigma_1, h_2 f^*\sigma_2)_p = h_1(p)h_2(p)g_E(\sigma_1,\sigma_2)_{f(p)}$$
$$\nabla_V^{f^*E} h_1 f^*\sigma_1 = V(h_1)f^*\sigma_1 + f^*(\nabla_V^E\sigma_1).$$

Finally, if M is a submanifold of a Riemannian manifold (X, g) then we point out that M inherits a metric from g by restricting it on M to vectors tangent to M.

2.5 E-valued forms

We defined a connection as a map

$$\nabla^E : \mathcal{C}^\infty(M;TM) \times \mathcal{C}^\infty(M;E) \longrightarrow \mathcal{C}^\infty(M;E)$$

but we can equivalently think of it as a map

$$\nabla^E : \mathcal{C}^\infty(M;E) \longrightarrow \mathcal{C}^\infty(M;T^*M) \times \mathcal{C}^\infty(M;E) = \mathcal{C}^\infty(M;T^*M \otimes E).$$

The space $\mathcal{C}^\infty(M; T^*M\otimes E)$ is the 1-form part of the space of **E-valued differential k-forms,**

$$\Omega^k(M;E) = \mathcal{C}^\infty(M;\Lambda^k T^*M \otimes E).$$

This space is linearly spanned by 'elementary tensors', i.e., section of the form $\omega\otimes s$ where $\omega \in \Omega^k(M)$ and $s \in \mathcal{C}^\infty(M;E)$. There is a wedge product between k-forms and E-valued ℓ-forms given on elementary tensors by

$$(\omega \otimes s) \wedge \eta = (\omega \wedge \eta) \otimes s$$

and then extended by linearity.

There is also a natural wedge product between a form in $\Omega^*(M; \operatorname{End} E)$ and a form in $\Omega^*(M;E)$ resulting in another form in $\Omega^*(M;E)$. For elementary tensors

$$\omega \otimes \Phi \in \Omega^*(M; \operatorname{End} E), \quad \eta \otimes s \in \Omega^*(M;E)$$

the wedge product is given by

$$(\omega \otimes \Phi) \wedge (\eta \otimes s) = (\omega \wedge \eta) \otimes \Phi(s).$$

Suppose M is Riemannian manifold, so that we have the Levi-Civita connection on TM, and that we have a connection on E, then as explained above there is an induced connection on $\Omega^k(M;E)$,

$$\nabla^{\Omega^k(M;E)} : \Omega^k(M;E) \longrightarrow \mathcal{C}^\infty(M; T^*M \otimes \Lambda^k T^*M \otimes E).$$

We can take the final section, 'antisymmetrize' all of the cotangent variables, and end up with a section of $\Omega^{k+1}(M;E)$. We think of the resulting map as an E-valued exterior derivative and denote it

$$d^E : \Omega^k(M;E) \longrightarrow \Omega^{k+1}(M;E).$$

One elementary tensors $\omega \otimes s$, d^E is given by

$$d^E(\omega \otimes s) = d\omega \otimes s + (-1)^{\deg(\omega)} \omega \wedge \nabla^E s$$

and in particular on $\Omega^0(M;E) = \mathcal{C}^\infty(M;E)$ coincides with ∇^E. Unlike the case $E = \mathbb{R}$, it is generally *not* true that $(d^E)^2 = 0$.

Proposition 2. *Let (M, g) be a Riemannian manifold, $E \longrightarrow M$ a vector bundle, and let d^E be the E-valued exterior derivative induced by a connection ∇^E on E and the Levi-Civita connection on M. The map*

$$(d^E)^2 : \Omega^k(M;E) \longrightarrow \Omega^{k+2}(M;E)$$

is given by wedge product with a differential form

$$R^E \in \Omega^2(M; \operatorname{End} E).$$

Moreover, if $V, W \in \mathcal{C}^\infty(M; TM)$, *we have*

$$R^E(V, W) = \nabla^E_V \nabla^E_W - \nabla^E_W \nabla^E_V - \nabla^E_{[V,W]}.$$

This 2-form with values in $\operatorname{End} E$ is is known as the **curvature of the connection** ∇^E. We say that the connection is **flat** if $R^E = 0$.

Proof. First consider the effect of applying ∇^E and then $\nabla^{T^*M\otimes E}$ to a section of E,

$$\mathcal{C}^\infty(M; E) \xrightarrow{\nabla^E} \mathcal{C}^\infty(M; T^*M \otimes E) \xrightarrow{\nabla^{T^*M\otimes E}} \mathcal{C}^\infty(M; T^*M \otimes T^*M \otimes E).$$

By the Leibnitz rule defining the connection on $T^*M \otimes E$, we have, for all $V, W \in \mathcal{C}^\infty(M; TM)$,

$$\begin{gathered} (\nabla^{T^*M\otimes E} \nabla^E \sigma)(V, W) = (\nabla^{T^*M\otimes E}_V (\nabla^E \sigma))(W) \\ = \nabla^E_V (\nabla^E_W \sigma) - (\nabla^E \sigma)(\nabla_V W) = \nabla^E_V (\nabla^E_W \sigma) - \nabla^E_{\nabla_V W} \sigma. \end{gathered}$$

We obtain d^E by anti-symmetrizing, so we have

$$\begin{aligned} (d^E)^2(\sigma)(V, W) &= \nabla^E_V (\nabla^E_W \sigma) - \nabla^E_{\nabla_V W} \sigma - \nabla^E_W (\nabla^E_V \sigma) + \nabla^E_{\nabla_W V} \sigma \\ &= \left(\nabla^E_V \nabla^E_W - \nabla^E_W \nabla^E_V - \nabla^E_{[V,W]} \right) \sigma. \end{aligned}$$

This is what we wanted to prove for $k = 0$, and the same computation replacing E with $\Lambda^k T^*M \otimes E$ proves the result for general k. □

The curvature of the Levi-Civita connection is referred to as the **Riemannian curvature** of (M, g).

2.6 Curvature

As explained in the previous section, the curvature of a Riemannian manifold (M, g) is $R \in \Omega^2(M; \operatorname{End}(TM))$, defined in terms of the Levi-Civita connection by

$$R(V, W) = \nabla_V \nabla_W - \nabla_W \nabla_V - \nabla_{[V,W]}.$$

Together with the obvious (anti)symmetry $R(V, W) = -R(W, V)$, we have the following:

Proposition 3. *Let (M, g) be a Riemannian manifold, and R the curvature of its Levi-Civita connection ∇, then for any $V, W, X, Y \in \mathcal{C}^\infty(M; TM)$ we have*

*a) (*First Bianchi Identity*)* $R(V, W)X + R(W, X)V + R(X, V)W = 0$

b) $g(R(V, W)X, Y) = -g(R(V, W)Y, X)$

c) $g(R(V, W)X, Y) = g(R(X, Y)V, W)$

Proof. a) reduces by a straightforward computation to the Jacobi identity for the commutator of vector fields.
b) is equivalent to proving that $g(R(V, W)X, X) = 0$, which we prove by noting that

$$\begin{gathered} g(\nabla_V \nabla_W X - \nabla_V \nabla_W X, X) \\ = (Vg(\nabla_W X, X) - g(\nabla_W X, \nabla_V X)) - (Wg(\nabla_V X, X) - g(\nabla_V X, \nabla_W X)) \\ = \frac{1}{2}[V, W]g(X, X) = g(\nabla_{[V,W]}X, X). \end{gathered}$$

c) follows from the other symmetries. Starting with following consequences of the Bianchi identity:

$$\begin{gathered} g(R(V, W)X + R(W, X)V + R(X, V)W, Y) = 0 \\ g(R(W, X)Y + R(X, Y)W + R(Y, W)X, V) = 0 \\ g(R(X, Y)V + R(Y, V)X + R(V, X)Y, W) = 0 \\ g(R(Y, V)W + R(V, W)Y + R(W, Y)V, X) = 0 \end{gathered}$$

we add all of these equations up, and use the other symmetries to see that we get

$$2g(R(X, V)W, Y) + 2g(R(Y, W)X, V) = 0$$

and hence $g(R(X, V)W, Y) = g(R(W, Y)X, V)$. □

In local coordinates $\{y_1, \ldots, y_m, \partial_{y_1}, \ldots, \partial_{y_m}\}$ on the tangent bundle, the curvature is determined by the functions R^ℓ_{ijk} defined by

$$R(\partial_i, \partial_j)\partial_k = R^\ell_{ijk}\partial_\ell.$$

We can compute these functions explicitly in terms of the Christoffel symbols of the Levi-Civita connection,

$$R^\ell_{ijk} = \Gamma^\ell_{is}\Gamma^s_{jk} - \Gamma^\ell_{js}\Gamma^s_{ik} + \partial_i(\Gamma^\ell_{jk}) - \partial_j(\Gamma^\ell_{ik}).$$

The Riemannian curvature vanishes if and only if (M, g) is locally isometric to Euclidean space [Spivak,vol 2].

There is a lot of complexity in the curvature tensor, so it is standard to focus on one of its contractions. Let $p \in M$ and $v, w \in T_pM$, the **Ricci curvature** at v, w, $\mathrm{Ric}(v, w)$, is the trace of the endomorphism of T_pM

$$z \mapsto R(v, z)w.$$

This defines a symmetric tensor $\mathrm{Ric} \in \mathcal{C}^\infty(M; T^*M \otimes T^*M)$. The metric is also a symmetric section of $T^*M \otimes T^*M$, so we can compare these two tensors. We say that a metric is *Einstein* if there is a constant λ such that

$$\mathrm{Ric} = \lambda g.$$

The Ricci curvature corresponds to a linear self-adjoint map

$$\widetilde{\mathrm{Ric}} : \mathcal{C}^\infty(M; TM) \longrightarrow \mathcal{C}^\infty(M; TM)$$

determined by

$$g(\widetilde{\mathrm{Ric}}(V), W) = \mathrm{Ric}(V, W).$$

The **scalar curvature** of a Riemannian manifold $\mathrm{scal} \in \mathcal{C}^\infty(M)$ is the trace of $\widetilde{\mathrm{Ric}}$.

Finally, the **sectional curvature** of a Riemannian manifold (M, g) is a function that assigns to each 2-dimensional subspace of T_pM a real number. Let $\Sigma \subseteq T_pM$ be a two dimensional space spanned by $v, w \in T_pM$ then

$$K(\Sigma) = -\left(\frac{g(R(v, w)v, w)}{g(v, v)g(w, w) - g(v, w)^2}\right)$$

is independent of the choice of v, w. Notice that the denominator is the area of the parallelogram spanned by v and w. The sectional curvature determines the full curvature tensor. A manifold is said to have **constant curvature** if the sectional curvature is constant. If the sectional curvature at a point $p \in M$ is constant and equal to K, then the Riemannian curvature at that point is given by

$$g(R(v, w)x, z)_p = K(g(w, x)_p g(v, z)_p - g(v, x)_p g(w, z)_p),$$

the Ricci curvature at p is $(m-1)Kg_p$ and the scalar curvature is $m(m-1)K$. It turns out that if this is true at each point of M, with *a priori* a different K at each point, then in fact all of the K's are equal and the manifold has constant curvature. The *simply connected* manifolds of constant curvature are, up to scale, just Euclidean space, the sphere, and hyperbolic space (see exercises 4 and 5).

In two dimensions, the curvature is determined by the scalar curvature which, up to a factor of 2, coincides with the classical Gaussian curvature. In three dimensions, the curvature is determined by the Ricci curvature.

In local coordinates $\{y_1, \ldots, y_m, \partial_{y_1}, \ldots, \partial_{y_m}\}$ on the tangent bundle, the Ricci curvature is determined by the functions Ric_{ij}

$$\mathrm{Ric}_{ij} = \mathrm{Ric}(\partial_i, \partial_j) = R^s_{isj}.$$

The map $\widetilde{\mathrm{Ric}}$ is given in local coordinates by

$$\widetilde{\mathrm{Ric}}(\partial_i) = (\widetilde{\mathrm{Ric}})^j_i \partial_j, \text{ with } (\widetilde{\mathrm{Ric}})^j_i = g^{js}\,\mathrm{Ric}_{is}$$

and hence the scalar curvature is given by

$$\mathrm{scal} = (\widetilde{\mathrm{Ric}})^k_k = g^{jk}\,\mathrm{Ric}_{jk} = g^{jk} R^s_{jsk}.$$

For the sake of intuition, recall what the sign of the Gaussian curvature tells us about a surface embedded in $\mathbb{R}^3$. Positive Gaussian curvature at a point means that the surface lies on a single side of its tangent plane at that point. Thus we can think of the sphere or a paraboloid. Negative Gaussian curvature at a point means that the surface locally looks like a saddle. Vanishing of the Gaussian curvature is true for a plane, but it is also true for a cylinder, since these are locally isometric. We should think of this as saying that there is at least one direction that looks Euclidean. For higher dimensional manifolds, these mental pictures are good descriptions of the behavior of sectional curvature. In fact, in the language of the next section, the sectional curvature of the space $\Sigma \subseteq T_pM$ is the Gaussian curvature of the surface in M carved out by geodesics through p in the directions in Σ, i.e., the image of Σ under the exponential map.

2.7 Geodesics

A curve $\gamma : [a, b] \longrightarrow M$ on a Riemannian manifold (M, g) is said to be a **geodesic** if, for all t,

$$\nabla_{\gamma'(t)}\gamma'(t) = 0.$$

In local coordinates, $\{y_1, \ldots, y_m, \partial_{y_1}, \ldots, \partial_{y_m}\}$, in which $\gamma(t) = (\gamma^1(t), \ldots, \gamma^m(t))$, this means that

$$\partial_t^2\gamma^k(t) + \Gamma^k_{ij}\partial_t\gamma^i(t)\partial_t\gamma^j(t) = 0. \tag{2.9}$$

This is a semi-linear second order system of ODEs for the coefficients $\gamma^i(t)$, so we have the following existence result[1]

[1] For more detail, see Lemma 10.2 of JOHN MILNOR, Morse theory. Annals of Mathematics Studies, No. 51 Princeton University Press, Princeton, N.J. 1963

Lemma 4. *For every point p_0 on a Riemannian manifold (M, g) there is a neighborhood $\mathcal{U}$ of p_0 and numbers $\varepsilon_1, \varepsilon_2 > 0$ such that: For all $p \in \mathcal{U}$ and tangent vector $v \in T_pM$ with $|v|_g < \varepsilon$ there is a unique geodesic*

$$\gamma_{(p,v)} : (-2\varepsilon_2, 2\varepsilon_2) \longrightarrow M,$$

satisfying the initial conditions

$$\gamma_{(p,v)}(0) = p, \quad \gamma'_{(p,v)}(0) = v. \tag{2.10}$$

Furthermore, the geodesics depend smoothly on the initial conditions.

The solution to the system (2.9) is unique up to its domain of definition. Since (2.9) is non-linear the geodesic may not be defined on all of $\mathbb{R}$, but rather there is a maximal interval around zero, say (a, b), in which this solution is defined. With this domain, the solution is unique, so we say that *for each* $(p, v) \in M$ *there is a unique maximally extends geodesic satisfying* (2.10). We will use $\gamma_{(p,v)}$ to refer to this maximally extended geodesic.

Remark 4. If M is a submanifold of $\mathbb{R}^N$ with the induced metric then a curve in M is a geodesic if and only if its acceleration as a curve in $\mathbb{R}^N$ is normal to M. In particular, if the intersection of M with a subspace of $\mathbb{R}^N$ that is everywhere orthogonal to M is a curve, then any constant speed parametrization of that curve must be a geodesic. The same is true replacing $\mathbb{R}^N$ with a Riemannian manifold, X. If $M \subseteq X$ is a Riemannian submanifold and all of the geodesics of M are also geodesics of X, we say that M is a *totally geodesic* submanifold.

By uniqueness we have the following homogeneity property

$$\gamma_{(p,\lambda v)}(t) = \gamma_{(p,v)}(\lambda t)$$

for all $\lambda \in \mathbb{R}$ for which one of the two sides is defined. Thus by making ε_1 smaller in the lemma, we can take, e.g., $\varepsilon_2 = 1$.

At each point $p \in M$ we can define **the exponential map** by

$$\exp_p : \mathcal{U} \subseteq T_pM \longrightarrow M, \quad \exp_p(v) = \gamma_{(p,v)}(1)$$

where $\mathcal{U}$ is the subset of T_pM of vectors v for which $\gamma_{(p,v)}(1)$ is defined. We know that $\mathcal{U}$ contains a neighborhood of the origin and that $\exp_p$ is smooth on $\mathcal{U}$.

Let $\{y_1, \ldots, y_m, \partial_{y_1}, \ldots, \partial_{y_m}\}$ be local coordinates for TM over a coordinate chart $\mathcal{V}$ of p. Consider the smooth function $F : \mathcal{U} \longrightarrow M \times M$, defined by

$$F(p, v) = (p, \exp_p(v)).$$

Notice that, if $\{y_1, \ldots, y_m, z_1, \ldots, z_m\}$ are the corresponding coordinates on $\mathcal{V}\times\mathcal{V} \subseteq M \times M$, then

$$DF : T\mathcal{U} \longrightarrow TM \times TM \text{ satisfies } DF(\partial_{y_i}) = \partial_{y_i} + \partial_{z_i}, \quad DF(\partial_{(\partial_{y_i})}) = \partial_{z_i}$$

and hence has the form $\begin{pmatrix} \mathrm{Id} & \mathrm{Id} \\ 0 & \mathrm{Id} \end{pmatrix}$. In particular, we can apply the implicit function theorem to F to conclude as follows:

Lemma 5. *For each $p \in M$ there is a neighborhood $\mathcal{U}$ of M and a number $\varepsilon > 0$ such that:*

a) Any two points $p, q \in \mathcal{U}$ are joined by a unique geodesic in M of length less than ε.

b) This geodesic depends smoothly on the two points.

c) For each $q \in \mathcal{U}$ the map $\exp_q$ maps the open ε-ball in T_qM diffeomorphically onto an open set $\mathcal{U}_q$ containing $\mathcal{U}$.

Since $\exp_p$ is a diffeomorphism from $B_\varepsilon(0) \subseteq T_pM$ into M, it defines a coordinate chart on M. These coordinates are known as **normal coordinates**.

At any point $p \in M$, one can show that if $v \in T_pM$ is small enough, the distance (2.4) between p and $\exp(v)$ is $|v|_g$. In particular $d(p, q) = 0 \implies p = q$ and hence (M, d) is a metric space.

A very nice property of a normal coordinate system is that near the origin the coefficients of the metric agree to second order with the identity matrix up to second order. More precisely[2]:

Proposition 6. *Let (M, g) be a Riemannian manifold, $p \in M$, and let $\{y_1, \ldots, y_m\}$ be normal coordinates centered at p. Then we have*

$$g_{ij}(0) = \delta_{ij}, \quad \Gamma^k_{ij}(0) = 0,$$
$$R_{ijk\ell}(0) = \frac{1}{2}\left[\frac{\partial^2 g_{j\ell}}{\partial y_i \partial y_k} - \frac{\partial^2 g_{jk}}{\partial y_i \partial y_\ell} + \frac{\partial^2 g_{ik}}{\partial y_j \partial y_\ell} - \frac{\partial^2 g_{i\ell}}{\partial y_j \partial y_k}\right]$$

and the Taylor expansion of the functions g_{ij} at the origin has the form

$$g_{ij}(y) \sim \delta_{ij} - \frac{1}{3}\sum_{k,\ell} R_{ikj\ell}\, y_k y_\ell + \mathcal{O}(|y|^3).$$

[2]See, e.g., Proposition 1.28 of Nicole Berline, Ezra Getzler, Michèle Vergne Heat kernels and Dirac operators. Corrected reprint of the 1992 original. Grundlehren Text Editions. Springer-Verlag, Berlin, 2004. x+363 pp. ISBN: 3-540-20062-2

2.8 Complete manifolds of bounded geometry

At each point p in a Riemannian radius, the injectivity radius at p, $\mathrm{inj}_M(p)$, is the largest radius R such that

$$\exp_p : B_R(0) \subseteq T_pM \longrightarrow M$$

is a diffeomorphism. The **injectivity radius** of M is defined by

$$\mathrm{inj}_M = \inf\{\mathrm{inj}_M(p) : p \in M\}.$$

By Lemma 5, a compact manifold has a positive injectivity radius.

We say that a manifold is **complete** if there is a point $p \in M$ for which $\exp_p$ is defined on all of T_pM (though it need not be injective on all of T_pM). The nomenclature comes from the Hopf-Rinow theorem, which states that the following assertions are equivalent:
a) The metric space (M, d), with d defined by (2.4), is complete.
b) Any closed and bounded subset of M is compact.
c) There is a point $p \in M$ for which $\exp_p$ is defined on all of T_pM.
d) At any point $p \in M$, $\exp_p$ is defined on all of T_pM.
Moreover in this case one can show that any two points $p, q \in M$ can be joined by a geodesic γ with $d(p, q) = \ell(\gamma)$.

A Riemannian manifold (M, g) is called a manifold with **bounded geometry** if:
1) M is complete and connected
2) $\mathrm{inj}_M > 0$
3) For every $k \in \mathbb{N}_0$, there is a constant C_k such that

$$|\nabla^k R|_g \leq C_k.$$

Every compact Riemannian manifold has bounded geometry. Many non-compact manifolds, such as $\mathbb{R}^m$ and $\mathbb{H}^m$, have bounded geometry. It is often the case that constructions that one can do on $\mathbb{R}^m$ can be carried out also on manifolds of bounded geometry but that one runs into trouble trying to carry them out on manifolds that do not have bounded geometry.

2.9 Riemannian volume and formal adjoint

On $\mathbb{R}^m$ with the Euclidean metric and the canonical orientation, there is a natural choice of volume form

$$\mathrm{dvol}_{\mathbb{R}^m} = dy_1 \wedge \cdots \wedge dy_m.$$

This assigns unit volume to the unit square. The volume of a parallelotope generated by m-vectors $z_1, \ldots, z_m$ is equal to the determinant of the change of basis matrix from y to z, which we can express as the 'Gram determinant'

$$\sqrt{\det(z_i \cdot x_j)}.$$

It turns out that a Riemannian metric g on an orientable manifold M also determines a natural volume form, dvol_g, determined by the property that, at any $p \in M$ and any local coordinates in which $g_{ij}(p) = \delta_{ij}$ we have $\mathrm{dvol}_g(p) = \mathrm{dvol}_{\mathbb{R}^m}$. In arbitrary oriented local coordinates, $x_1, \ldots, x_m$, $\mathrm{dvol}(p)$ should be the volume of the parallelotope generated by ∂_{x_i} and hence

$$\mathrm{dvol}_g(p) = \sqrt{\det(g_{ij}(p))}\, dx_1 \wedge \cdots \wedge dx_m.$$

The volume form is naturally associated to the metric in that, e.g.,

$$\mathrm{dvol}_{f^*g} = f^* \,\mathrm{dvol}_g$$

and one can check that the volume form is covariantly constant with respect to the Levi-Civita connection.

Given a metric g_E on a bundle $E \longrightarrow M$ and a Riemannian metric on M, we obtain an inner product on sections of E by setting:

$$\langle \cdot, \cdot \rangle_E : \mathcal{C}^\infty(M; E) \times \mathcal{C}^\infty(M; E) \longrightarrow \mathbb{R},$$
$$\langle \sigma, \tau \rangle_E = \int_M g_E(\sigma, \tau)\, \mathrm{dvol}_g, \quad \text{for all } \sigma, \tau \in \mathcal{C}^\infty(M; E).$$

This is a non-degenerate, positive definite inner product on $\mathcal{C}^\infty(M; E)$ because g_E is a non-degenerate positive definite inner product on each E_p.

Using this inner product, we can define the L^p-norms in the usual way

$$\|\cdot\|_{L^p(M;E)} : \mathcal{C}^\infty(M; E) \longrightarrow \mathbb{R}, \quad \|s\|_{L^p(M;E)} = \left(\int_M |s|^p_{g_E}\, \mathrm{dvol}_g \right)^{1/p}$$

and then define the L^p **space of sections of** E, $L^p(M; E)$, by taking the $\|\cdot\|_{L^p(M;E)}$-closure of the smooth functions with finite $L^p(M; E)$-norm. These are Banach spaces, and we will mostly work with $L^2(M; E)$, which is also a Hilbert space.

If we have a Riemannian manifold (M, g) and two vector bundles $E \longrightarrow M$, $F \longrightarrow M$ with bundle metrics, we say that two linear operators

$$T : \mathcal{C}^\infty(M; E) \longrightarrow \mathcal{C}^\infty(M; F), \quad T^* : \mathcal{C}^\infty(M; F) \longrightarrow \mathcal{C}^\infty(M; E)$$

are **formal adjoints** if they satisfy

$$\langle T\sigma, \tau\rangle_F = \langle \sigma, T^*\tau\rangle_E, \quad \text{for all } \sigma \in \mathcal{C}_c^\infty(M;E), \tau \in \mathcal{C}_c^\infty(M;F).$$

(We use the word formal because we are not working with a complete vector space like L^2 or L^p but just with smooth functions, this avoids some technicalities.) As a simple example, the formal adjoint of a scalar differential operator on $\mathbb{R}$ is easy to compute

$$\left(\sum a_k(x)\partial_x^k\right)^* u(x) = \sum(-1)^k\partial_x^k(a_k(x)u(x)), \text{ for all } u \in \mathcal{C}^\infty(\mathbb{R}).$$

The formal adjoint of any differential operator is similarly computed using integration by parts in local coordinates (the compact supports allow us not to worry about picking up 'boundary terms').

We say that a linear operator is **formally self-adjoint** if it coincides with its formal adjoint. For instance, if T and T^* are formal adjoints then TT^* and T^*T are formally self-adjoint. Moreover, these operators are positive semidefinite since on compactly supported sections

$$\langle T^*Ts, s\rangle = \langle Ts, Ts\rangle = |Ts|^2.$$

In the next section we will compute the formal adjoints of d and ∇^E, which will lead to natural Laplace-type operators.

2.10 Hodge star and Hodge Laplacian

We have already looked at the space of differential forms on M and seen that the smooth structure on M induces a lot of structure on $\Omega^*(M)$, such as the exterior product, interior product and exterior derivative. Now we suppose that (M, g) is a Riemannian manifold and we will see that there is still more structure on $\Omega^*(M)$.

First, we have already noted above that a metric on M induces metrics on associated vector bundles, such as Λ^*T^*M. It is easy to describe this metric in terms of the metric on T^*M when dealing with elementary k-forms, namely

$$g_{\Lambda^k T^*M}(\omega^1 \wedge \cdots \wedge \omega^k, \theta^1 \wedge \cdots \theta^k)_p = \det\left(g_{T^*M}(\omega^i, \theta^j)_p\right), \quad \text{for all } \{\omega^i, \theta^j\} \subseteq T_p^*M.$$

On any oriented Riemannian m-manifold, we define the **Hodge star operator**

$$* : \Omega^k(M) \longrightarrow \Omega^{m-k}(M)$$

by demanding, for all $\omega, \eta \in \Omega^k(M)$,

$$\omega \wedge *\eta = g_{\Lambda^k T^*M}(\omega, \eta)\, \mathrm{dvol}_g\,.$$

Thus, for example, in $\mathbb{R}^4$ with Euclidean metric and canonical orientation, with $y_1, \ldots, y_4$ an orthonormal basis,

$$\begin{gathered} *1 = dy^1 \wedge dy^2 \wedge dy^3 \wedge dy^4 \\ *dy^1 = dy^2 \wedge dy^3 \wedge dy^4, \quad *dy^2 = -dy^1 \wedge dy^3 \wedge dy^4, \\ *dy^1 \wedge dy^2 = dy^3 \wedge dy^4, \quad *dy^2 \wedge dy^4 = -dy^1 \wedge dy^3 \\ *dy^1 \wedge dy^2 \wedge dy^3 = dy^4, \quad *dy^1 \wedge dy^2 \wedge dy^3 \wedge dy^4 = 1 \end{gathered}$$

and so on. Note that $*$ is symmetric in that

$$\omega \wedge *\eta = g_{\Lambda^k T^*M}(\omega, \eta)\, \mathrm{dvol}_g = g_{\Lambda^k T^*M}(\eta, \omega)\, \mathrm{dvol}_g = \eta \wedge *\omega.$$

The Hodge star is almost an involution,

$$\omega \in \Omega^k(M) \implies *(*\omega) = (-1)^{k(m-k)}\omega.$$

The Hodge star acts on differential forms and we have

$$\langle \omega, \eta \rangle_{\Lambda^* T^*M} = \int_M \omega \wedge *\eta.$$

This lets us get a formula for the **formal adjoint of the exterior derivative,** $d^* = \delta$. Namely, whenever $\omega \in \Omega_c^{k-1}(M)$, and $\eta \in \Omega_c^k(M)$,

$$\begin{aligned} \langle d\omega, \eta \rangle_{\Lambda^k T^*M} &= \int_M (d\omega) \wedge *\eta = \int_M \big(d(\omega \wedge *\eta) - (-1)^{k-1}\omega \wedge d*\eta\big) \\ &= (-1)^k \int_M \omega \wedge d*\eta = (-1)^k \int_M \omega \wedge (-1)^{(m-k+1)(k-1)} *^2 d*\eta \\ &= \langle \omega, (-1)^{m(k+1)+1} * d*\eta \rangle_{\Lambda^{k-1}T^*M} \end{aligned}$$

and so, when acting on $\Omega^k(M)$,

$$\delta = d^* = (-1)^{m(k+1)+1} * d* = (-1)^k *^{-1} d*\,.$$

A form is called **coclosed** if it is in the null space of δ. (Functions are thus automatically coclosed.)

In particular on one-forms we have

$$\omega \in \Omega^1(M) \implies \delta\omega = -*d*\omega,$$

and, if $\omega \in \Omega^1_c(M)$, then Stokes' theorem (on manifolds with boundary) says that

$$\int_M \delta(\omega)\,\mathrm{dvol}_g = -\int_M *d*\omega\,\mathrm{dvol}_g = -\int_M d(*\omega) = -\int_{\partial M} *\omega.$$

Hence on manifolds without boundary,

$$\int_M \delta(\omega)\,\mathrm{dvol}_g = 0.$$

The **Hodge Laplacian on k-forms** is the differential operator

$$\Delta_k : \Omega^k(M) \longrightarrow \Omega^k(M), \quad \Delta_k = d\delta + \delta d.$$

Note that this operator is formally self-adjoint. Elements of the null space of the Hodge Laplacian are called **harmonic differential forms**, and we will use the notation

$$\mathcal{H}^k(M) = \{\omega \in \Omega^k(M) : \Delta_k\omega = 0\}$$

for the vector space of harmonic k-forms.

A form that is closed and coclosed is automatically harmonic. Conversely, if $\omega \in \Omega^k_c(M)$ is compactly supported and harmonic, then

$$0 = \langle \Delta\omega, \omega\rangle_{\Lambda^k T^*M} = \langle d\omega, d\omega\rangle_{\Lambda^{k+1}T^*M} + \langle \delta\omega, \delta\omega\rangle_{\Lambda^{k-1}T^*M}$$

and so ω is both closed and coclosed. In particular, if M is compact, we have a natural map

$$\mathcal{H}^k(M) \longrightarrow \mathrm{H}^k_{dR}(M)$$

since each harmonic form is closed. We will show later that this map is an isomorphism by showing that every de Rham cohomology class on a compact manifold contains a unique harmonic form.

Since $d^2 = 0$ and $\delta^2 = 0$, we can also write the Hodge Laplacian as

$$\Delta_k = (d+\delta)^2.$$

Also, notice that

$$*\Delta_k = \Delta_{m-k}*,$$
$$d\Delta_k = d\delta d = \Delta_{k+1}d, \quad \delta\Delta_k = \delta d\delta = \Delta_{k-1}\delta$$

and hence ω is harmonic implies $d\omega$ and $\delta\omega$ harmonic, and is equivalent to $*\omega$ harmonic. In particular,

$$\mathcal{H}^k(M) = \mathcal{H}^{m-k}(M),$$

which we call *Poincaré duality for harmonic forms.*

Let us describe the Laplacian on functions $\Delta = \Delta_0$ in a little more detail. Since δ vanishes on functions, we have

$$\Delta = \delta d = - * d * d.$$

A closed function is necessarily a constant, so on compact manifolds all harmonic functions are constant. In local coordinates, $\{y_1, \ldots, y_m\}$, the Laplacian is given by

$$\Delta f = -\frac{1}{\sqrt{\det g_{ij}}} \frac{\partial}{\partial y_j} \left(\sqrt{\det g_{ij}}\, g^{ij} \frac{\partial f}{\partial y_i} \right) = - \left(g^{ij} \partial_i \partial_j - g^{ij} \Gamma_{ij}^k \partial_k \right) f.$$

On Euclidean space our Laplacian is

$$\Delta = - \sum_{i=1}^m \frac{\partial^2 f}{\partial y_i^2}$$

which is, up to a sign, the usual Laplacian on $\mathbb{R}^m$. With our sign convention, the Laplacian is sometimes called the 'geometer's Laplacian' while the opposite sign convention yields the 'analyst's Laplacian'. The Laplacian on functions on a Riemannian manifold is sometimes called the Laplace-Beltrami operator to emphasize that there is a Riemannian metric involved.

2.11 Exercises

Exercise 1.
Show that if $(M_1.g_1)$ and (M_2, g_2) are Riemannian manifolds, then so is $(M_1 \times M_2, g_1 + g_2)$.

Exercise 2.
It is tempting to think that the interior product and the exterior derivative will define a covariant derivative on T^*M without having to use a metric, say by

$$\widetilde{\nabla}_V \omega = V \lrcorner\, d\omega, \text{ for all } \omega \in \mathcal{C}^\infty(M; T^*M), V \in \mathcal{C}^\infty(M; TM).$$

Show that this is *not* a covariant derivative on T^*M.

Show that the exterior derivative does define a connection on the trivial bundle $\mathbb{R} \times M \longrightarrow M$.

Exercise 3.
Show that parallel transport determines the connection.

Exercise 4.
Consider the 2-dimensional sphere of radius r centered at the origin of $\mathbb{R}^3$,

$$\mathbb{S}^2_r = \{(y_1, y_2, y_3) \in \mathbb{R}^3 : |y| = r\}.$$

The Euclidean metric on $\mathbb{R}^3$ induces the 'round' metric on $\mathbb{S}^2_r$. Compute the sectional curvatures of this metric.
Hint: Use the spherical coordinates (θ, ψ) that parametrize the sphere via

$$\phi(\theta, \phi) = (r \cos\theta \cos\psi, r\cos\theta \sin\phi, r\sin\theta).$$

Notice that any curve on $\mathbb{S}^2$ can be parametrized $\gamma(t) = (\theta(t), \psi(t))$ and the length of $\gamma'(t)$ is

$$r\sqrt{(\theta')^2 + (\psi')^2 \cos^2\theta}.$$

Hence the round metric in these coordinates is $g_{\mathbb{S}^2_r} = r^2 d\theta^2 + r^2 \cos^2\theta d\psi^2$.

Exercise 5.
One model of hyperbolic space of dimension m, $(\mathbb{H}^m, g_{\mathbb{H}^m})$ is the space

$$\mathbb{R}^m_+ = \{(x, y_1, \ldots, y_{m-1}) \in \mathbb{R}^m : x > 0\}$$

with the metric

$$g_{\mathbb{H}^m} = \frac{dx^2 + |dy|^2}{x^2}.$$

Compute the Christoffel symbols and the sectional curvatures of $\mathbb{H}^m$.

Exercise 6.
(Riemann's formula) Let K be a constant. Show that a neighborhood of the origin in $\mathbb{R}^m$ with coordinates $\{y_1, \ldots, y_m\}$ endowed with the metric

$$\frac{|dy|^2}{\left(1 + \frac{K}{4}|y|^2\right)^2}$$

has constant curvature K.

Exercise 7.
Use stereographic projection to write the metric on the sphere as a particular case of Riemann's formula.

Exercise 8.
Consider the flat torus $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$ with the metric it inherits from $\mathbb{R}^m$. Compute the curvature and geodesics of $\mathbb{T}^m$.

Exercise 9.
If $\gamma(t)$ is a geodesic on a Riemannian manifold show that $\gamma'(t)$ has constant length.

Exercise 10.
a) Determine the geodesics on the sphere. (Hint: Remark 4.)
b) All of the points of the sphere have the same injectivity radius, what is it?
c) Consider the geodesic triangle on the sphere $\mathbb{S}^2$ with a vertex at the 'north pole' and other vertices on the 'equator', suppose that at the north pole the edge form an angle θ. Describe the parallel translation map along this triangle.
Hint: Because the sides are geodesics, any vector parallel along the sides forms a constant angle with the tangent vector of the edge.

Exercise 11.
Give an example of two curves on a Riemannian manifold with the same endpoints and with different parallel transport maps.

Exercise 12.
a) Show that any differential k-form α on $M \times [0,1]$ has a 'standard decomposition'

$$\alpha = \alpha_t(s) + ds \wedge \alpha_n(s)$$

where ds is the unique positively oriented unit 1-form on $[0,1]$ and, for each $s \in [0,1]$, $\alpha_t(s)$ is a differential k-form on M and $\alpha_n(s)$ is a differential $(k-1)$-form on M. What is the standard decomposition of $d\alpha$ in terms of α_t and α_n?
b) For each $s \in [0,1]$ let

$$i_s : M \longrightarrow M \times [0,1], \quad i_s(p) = (p, s).$$

We can think of the pull-back of a differential form by i_s as the 'restriction of a form on $M \times [0,1]$ to $M \times \{s\}$'. Consider the map π_* that takes k-forms on $M \times [0,1]$ to $(k-1)$-forms on M given by

$$\pi_*(\alpha) = \int_0^1 \alpha_n(s)\, ds.$$

Show that for any differential k-form α on $M \times [0,1]$

$$i_1^*\alpha - i_0^*\alpha = d\pi_*\alpha + \pi_*(d\alpha).$$

c) Show that whenever $[0,1] \ni t \mapsto \omega_t$ is a smooth family of closed k-forms on a manifold M, the de Rham cohomology class of ω_t is independent of t.

Exercise 13.
(Poincaré's Lemma)
A manifold M is (smoothly) contractible if there is a point $p_0 \in M$, and a smooth map $H : M \times [0,1] \longrightarrow M$ such that

$$H(p,1) = p, \quad H(p,0) = p_0 \text{ for all } p \in M.$$

H is known as a 'contraction' of M.
a) Show that $\mathbb{R}^n$ is contractible and any manifold is locally contractible (i.e, every point has a contractible neighborhood).
b) Let M be a contractible smooth n-dimensional manifold, and let ω be a closed differential k-form on M (i.e., $d\omega = 0$). Prove that there is a $(k-1)$-form η on M such that $d\eta = \omega$.
(Hint: Let H be a contraction of M, and let $\alpha = H^*\omega$. Then $\omega = (H \circ i_1)^*\omega = i_1^*\alpha$ and $i_0^*\alpha = 0$. Check that $\eta = \pi_*\alpha$ works.)

Exercise 14.
Let M be an n-dimensional manifold with boundary ∂M. A (smooth) retraction of M onto its boundary is a smooth map $\phi : M \longrightarrow \partial M$ such that

$$\phi(\zeta) = \zeta, \text{ whenever } \zeta \in \partial M.$$

Prove that if M is compact and oriented, then there do not exist any retractions of M onto its boundary.
(Hint: Let ω be an $(n-1)$-form on ∂M with $\int_{\partial M} \omega > 0$ (how do we know there is such an ω?), notice that $d\omega = 0$ (why?). Assume that ϕ is a retraction of M onto its boundary, and apply Stokes' theorem to $\phi^*\omega$ to get a contradiction.)
If $\phi : M \longrightarrow \partial M$ is a continuous retraction, one can approximate it by a smooth retraction, so there are in fact no continuous retractions from M to ∂M.

Exercise 15.
(Brouwer fixed point theorem)
Let $\overline{B}$ be the closed unit ball in $\mathbb{R}^n$, show that any continuous map $F : B \longrightarrow B$ has a fixed point (i.e., there is a point $\zeta \in B$ such that $F(\zeta) = \zeta$).
(Hint: If $F : B \longrightarrow B$ is a smooth map without any fixed points, let $\phi(\zeta)$ be the point on ∂B where the ray starting at $F(\zeta)$ and passing through ζ hits the boundary. Show that ϕ is continuous and then apply the continuous version of the previous exercise.)

Exercise 16.
Let V_i be a local orthonormal frame for TM with dual coframe θ^i. Show that

$$d = \sum \mathfrak{e}_{\theta^i} \nabla_{V_i}, \quad \delta = -\sum \mathfrak{i}_{V_i} \nabla_{V_i}.$$

2.12 Bibliography

My favorite book for Riemannian geometry is the eponymous book by GALLOT, HULIN, and LAFONTAINE. Some other very good books are *Riemannian geometry and geometric analysis* by JOST, and *Riemannian geometry* by PETERSEN.

Perhaps the best place to read about connections is the second volume of Spivak, *A comprehensive introduction to differential geometry* which includes translations of articles by Gauss and Riemann. Spivak does a great job of presenting the various approaches to connections and curvature. In each approach, he explains how to view the curvature as the obstruction to flatness.

An excellent concise introduction to Riemannian geometry, which we follow when discussing geodesics, can be found in Milnor, *Morse theory.* Another excellent treatment is in volume one of Taylor, *Partial differential equations.*

Lecture 3

Laplace-type and Dirac-type operators

3.1 Principal symbol of a differential operator

Let M be a smooth manifold, $E \longrightarrow M$, $F \longrightarrow M$ real vector bundles over M, and D a differential operator of order ℓ acting between sections of E and sections of F, $D \in \mathrm{Diff}^\ell(M; E, F)$. In any coordinate chart $\mathcal{V}$ of M that trivializes both E and F, D is given by

$$D = \sum_{|\alpha| \leq \ell} a_\alpha(y) \partial_y^\alpha$$

where $\{y_1, \ldots, y_m\}$ are the coordinates on $\mathcal{V}$ and $a_\alpha \in \mathcal{C}^\infty(\mathcal{V}; \mathrm{Hom}(E, F)\big|_{\mathcal{V}})$.

This expression for D changes quite a bit if we choose a different set of coordinates. However it turns out we can make invariant sense of the 'top order part' of D,

$$\sum_{|\alpha| = \ell} a_\alpha(y) \partial_y^\alpha,$$

as a constant coefficient differential operator on each fiber of the tangent space of M. It is more common to take the dual approach and interpret this as a function on the cotangent bundle of M as follows: Let $\{y_1, \ldots, y_m, dy^1, \ldots, dy^m\}$ be the induced coordinates on the cotangent bundle of M, $\pi : T^*M \longrightarrow M$, over $\mathcal{V}$ and define **the principal symbol of** D at the point $\xi = \xi_j dy^j$ of the cotangent bundle, to be

$$\sigma_\ell(D)(y, \xi) = i^\ell \sum_{|\alpha| = \ell} a_\alpha(y) \xi^\alpha \in \mathcal{C}^\infty(T^*M; \pi^* \mathrm{Hom}(E, F)).$$

That is, we just ignore all the terms in the expression for the differential operator of order less than ℓ and replace differentiation with a polynomial in the cotangent variables.

Why the factor of i? A key fact that we will exploit later is that the Fourier transform on $\mathbb{R}^n$ interchanges differentiation by y_j with multiplication by $i\xi_j$,

$$\int_{\mathbb{R}^n} e^{-iy\cdot\xi}\partial_{y_j}f(y)\,dy = \int_{\mathbb{R}^n} e^{-iy\cdot\xi}(i\xi_j)f(y)\,dy.$$

Of course here one should be working with complex valued functions, and soon we will be. For the moment, just think of the power of i as a formal factor.

Another way of defining the principal symbol, which has the advantage of being obviously independent of coordinates, is: At each $\xi \in T_p^*M$, choose a function f such that $df(p) = \xi$ and then set

$$\sigma_\ell(D)(p,\xi) = \lim_{t\to\infty} t^{-\ell}\left(e^{-itf}\circ D\circ e^{itf}\right)(p) \in \operatorname{Hom}(E_p, F_p). \tag{3.1}$$

By the Leibnitz rule, $e^{-itf}\circ D\circ e^{-itf}$ is a polynomial in t of order ℓ and the top order term at p is of the form

$$(it)^\ell \sum_{|\alpha|=\ell} a_\alpha(p)(\partial_{y_1}^{\alpha_1}f)(p)\cdots(\partial_{y_m}^{\alpha_m}f)(p) = (it)^\ell \sum_{|\alpha|=\ell} a_\alpha(p)\xi_1^{\alpha_1}\cdots\xi_m^{\alpha_m}$$

which shows that (3.1) is independent of the choice of f and hence well-defined. Notice that the principal symbol of a differential operator is not just a smooth function on T^*M, but in fact a homogeneous polynomial of order ℓ on each fiber of T^*M, we will denote these functions by

$$\mathscr{P}_\ell(T^*M;\pi^*\operatorname{Hom}(E,F)) \subseteq \mathcal{C}^\infty(T^*M;\pi^*\operatorname{Hom}(E,F)).$$

For an operator of order one or two, it is easy to carry out the differentiations in (3.1) and see that

$$\sigma_1(D)(p,\xi) = \frac{1}{i}\left[D,f\right], \quad \sigma_2(D)(p,\xi) = -\frac{1}{2}\left[\left[D,f\right],f\right]$$

for any f such that $df(p) = \xi$. A similar formula holds for higher order operators.

The principal symbol of order ℓ vanishes if and only if the operator is actually of order $\ell - 1$, so we have a short exact sequence

$$0 \longrightarrow \operatorname{Diff}^{\ell-1}(M;E,F) \longrightarrow \operatorname{Diff}^{\ell}(M;E,F) \xrightarrow{\sigma_\ell} \mathscr{P}_\ell(T^*M;\pi^*\operatorname{Hom}(E,F)) \longrightarrow 0.$$

One very important property of the principal symbol map is that it is a homomorphism. That is, if

$$D_1 : \mathcal{C}^\infty(M;E_1) \longrightarrow \mathcal{C}^\infty(M;E_2), \quad D_2 : \mathcal{C}^\infty(M;E_2) \longrightarrow \mathcal{C}^\infty(M;E_3)$$

are differential operators of orders ℓ_1 and ℓ_2 respectively, then

$$D_2 \circ D_1 : \mathcal{C}^\infty(M; E_1) \longrightarrow \mathcal{C}^\infty(M; E_3)$$

is a differential operator with symbol

$$\sigma_{\ell_1+\ell_2}(D_2 \circ D_1)(p, \xi) = \sigma_{\ell_2}(D_2)(p, \xi) \circ \sigma_{\ell_1}(D_1)(p, \xi),$$

as follows directly from (3.1).

This is especially useful for scalar operators $E_i = \underline{\mathbb{R}}$ since then multiplication in $\mathscr{P}_{\ell_1+\ell_2}(T^*M)$ is commutative, so we always have

$$\sigma_{\ell_1}(D_1)(p, \xi) \circ \sigma_{\ell_2}(D_2)(p, \xi) = \sigma_{\ell_2}(D_2)(p, \xi) \circ \sigma_{\ell_1}(D_1)(p, \xi),$$

even though $D_1 \circ D_2$ is generally different from $D_2 \circ D_1$.

It is also true that the symbol map respects adjoints in that, if D and D^* are formal adjoints, then

$$\sigma_\ell(D^*) = \sigma_\ell(D)^*$$

where on the right we take the the adjoint in Hom, so in local coordinates just the matrix transpose.

We can use (3.1) to find the symbols of d and δ by noting that

$$[d, f]\,\omega = d(f\omega) - f d\omega = \mathfrak{e}_{df}\omega,$$
$$[\delta, f]\,\omega = \delta(f\omega) - f\delta\omega = -\mathfrak{i}_{df^\sharp}\omega$$

hence

$$\sigma_1(d)(p, \xi) = -i\mathfrak{e}_\xi : \Lambda^k T_p^*M \longrightarrow \Lambda^{k+1}T_p^*M$$
$$\sigma_1(\delta)(p, \xi) = i\mathfrak{i}_{\xi^\sharp} : \Lambda^k T_p^*M \longrightarrow \Lambda^{k-1}T_p^*M.$$

Then the symbol of the Hodge Laplacian is

$$\sigma_2(\Delta_k)(p, \xi) = (\sigma_1(d)\sigma_1(\delta) + \sigma_1(\delta)\sigma_1(d))\,(p, \xi) = \mathfrak{e}_\xi\mathfrak{i}_{\xi^\sharp} + \mathfrak{i}_{\xi^\sharp}\mathfrak{e}_\xi = |\xi|_g^2 \tag{3.2}$$

i.e., it is given by multiplying vectors in $\Lambda^k T_p^*M$ by the scalar $|\xi|_g^2$.

A differential operator is called **elliptic** if $\sigma_\ell(D)(p, \xi)$ is invertible at all $\xi \neq 0$. Hence Δ_k is an elliptic differential operator, d and δ are not elliptic, and $d \pm \delta$ are elliptic. The main aim of this course is to understand elliptic operators.

One particularly interesting property of an elliptic operator D on a compact manifold is that its 'index'

$$\operatorname{ind} D = \dim\ker D - \dim\ker D^*$$

is well-defined (in that both terms in the difference are finite) and remarkably stable, as we will show below. Before doing that, however, we will start by constructing examples of elliptic operators.

3.2 Laplace-type operators

A second-order differential operator on a $\mathbb{R}^k$-bundle $E \longrightarrow M$ over a Riemannian manifold (M, g) is of **Laplace-type** if its principal symbol at $\xi \in T_p^*M$ is scalar multiplication by $|\xi|_g^2$. Thus by (3.2) the Hodge Laplacian Δ_k is, for every k, a Laplace-type operator.

A general Laplace-type operator H can be written in any local coordinates as

$$H = -g^{ij}\partial_{y_i}\partial_{y_j} + a^k\partial_{y_k} + b, \text{ with } a^k, b \in \mathcal{C}^\infty(\mathcal{V}; \mathrm{Hom}(E, E)).$$

If $E \longrightarrow M$ is a vector bundle with connection ∇^E, then we can form the composition

$$\mathcal{C}^\infty(M; E) \xrightarrow{\nabla^E} \mathcal{C}^\infty(M; T^*M \otimes E) \xrightarrow{\nabla^{T^*M\otimes E}} \mathcal{C}^\infty(M; T^*M \otimes T^*M \otimes E)$$

and then take the trace over the two factors of T^*M to define the **connection Laplacian**,

$$\Delta^\nabla = -\operatorname{Tr}(\nabla^{T^*M\otimes E}\nabla^E) : \mathcal{C}^\infty(M; E) \longrightarrow \mathcal{C}^\infty(M; E).$$

(This is also known as the **Bochner Laplacian**.) Explicitly, we have

$$(\nabla^{T^*M\otimes E}\nabla^E s)(V, W) = \left(\nabla^E_V\nabla^E_W - \nabla^E_{\nabla_V W}\right) s$$

and so in local coordinates we have

$$\Delta^\nabla s = -\left(g^{ij}\nabla^E_{\partial_{y_i}}\nabla^E_{\partial_{y_j}} - g^{ij}\Gamma^k_{ij}\nabla^E_{\partial_{y_k}}\right) s.$$

Thus this is a Laplace-type operator.

Directly from the formula in local coordinates we see that the Laplacian on functions is the connection Laplacian for the Levi-Civita connection $\Delta^\nabla = \Delta$. It turns out that any Laplace-type operator on a bundle $E \longrightarrow M$ is equal to a connection Laplacian plus a zero-th order differential operator[1].

Proposition 7. *Let (M, g) be a Riemannian manifold, $E \longrightarrow M$ a vector bundle, and H a Laplace-type operator on sections of E. There exists a unique connection ∇^E on E and a unique section $Q \in \mathcal{C}^\infty(M; \mathrm{Hom}(E))$ such that*

$$H = \Delta^\nabla + Q.$$

[1]For the proof of this proposition see, e.g., Proposition 2.5 of Nicole Berline, Ezra Getzler, Michèle Vergne Heat kernels and Dirac operators. Corrected reprint of the 1992 original. Grundlehren Text Editions. Springer-Verlag, Berlin, 2004. x+363 pp. ISBN: 3-540-20062-2

3.3 Dirac-type operators

The physicist P.A.M. Dirac constructed first-order differential operators whose squares were Laplace-type operators (albeit in 'Lorentzian signature') for the purpose of extending quantum mechanics to the relativistic setting. After these operators were rediscovered by Atiyah and Singer, they were christened Dirac operators and later generalized to Dirac-type operators.

Let $E \longrightarrow M$ and $F \longrightarrow M$ be $\mathbb{R}^k$-vector bundles over a Riemannian manifold (M, g). A first order differential operator $D \in \mathrm{Diff}^1(M; E, F)$ is a **Dirac-type operator** if D^*D and DD^* are Laplace-type operators. If $E = F$ and $D = D^*$, we say that D is a **symmetric Dirac-type operator.** If $D \in \mathrm{Diff}^1(M; E, F)$ is a Dirac-type operator, then

$$\widetilde{D} = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix} \in \mathrm{Diff}^1(M; E \oplus F)$$

is a symmetric Dirac-type operator.

We have seen one example of a symmetric Dirac-type operator so far, the **de Rham operator**:

$$d + \delta \in \mathrm{Diff}^1(M; \Omega^*(M)), \text{ where } \Omega^*(M) = \bigoplus_{k=0}^{m} \Omega^k(M) = \mathcal{C}^\infty(M; \Lambda^* T^* M)$$

Indeed, we know that $(d + \delta)^* = d + \delta$ and $(d + \delta)^2 = \Delta$, the Hodge Laplacian on forms of all degrees.

We can use the symbol of an arbitrary symmetric Dirac operator D to uncover some algebraic structure on E. For every $p \in M$, and $\xi \in T_p M$, let

$$\theta(p, \xi) = \sigma_1(D)(p, \xi) : E_p \longrightarrow E_p,$$

and, when the point p is fixed, let $\theta(\xi) = \theta(p, \xi)$. Since D is symmetric, we have

$$\theta(\xi) = \theta(\xi)^*$$

and since it is a Dirac-type operator, we have

$$\theta(\xi)^2 = g(\xi, \xi)\,\mathrm{Id}\,.$$

We can 'polarize' this identity by applying it to $\xi + \eta \in T_p M^*$,

$$\begin{aligned} \theta(\xi + \eta)^2 &= \theta(\xi)^2 + \theta(\xi)\theta(\eta) + \theta(\eta)\theta(\xi) + \theta(\eta)^2 \\ &= g(\xi + \eta, \xi + \eta)\,\mathrm{Id} = (g(\xi, \xi) + 2g(\xi, \eta) + g(\eta, \eta))\,\mathrm{Id} \end{aligned}$$

and hence

$$\theta(\xi)\theta(\eta) + \theta(\eta)\theta(\xi) = 2g(\xi, \eta)\,\mathrm{Id}\,.$$

More generally, suppose we have two vector spaces V and W, a quadratic form q on V, and a linear map

$$\mathcal{cl} : V \longrightarrow \mathrm{End}(W)$$

such that[2]

$$\mathcal{cl}\,(v)\,\mathcal{cl}\,(v) = q(v)\,\mathrm{Id}\,. \tag{3.3}$$

We call $\mathcal{cl}$ **Clifford multiplication by elements of** V.

As above, we can polarize (3.3). Define $q(v, w) = \frac{1}{2}(q(v + w) - q(v) - q(w))$ and apply (3.3) to $v + w$ to obtain

$$\mathcal{cl}\,(v)\,\mathcal{cl}\,(w) + \mathcal{cl}\,(w)\,\mathcal{cl}\,(v) = 2q(v, w)\,\mathrm{Id}\,. \tag{3.4}$$

The map $\mathcal{cl}$ canonically extends to the **Clifford algebra of** (V, q), which is defined as follows. First let

$$\bigotimes V = \mathbb{R} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \dots$$

and note that the map $\mathcal{cl}$ extends to a map $\widehat{\mathcal{cl}} : \bigotimes V \longrightarrow \mathrm{End}(W)$. This extension vanishes on elements of the form

$$v \otimes w + w \otimes v - 2q(v, w)\,\mathrm{Id}$$

and hence on the ideal generated by these elements

$$\mathcal{I}_{\mathcal{cl}}(V) = \mathrm{span}\left\{\kappa \otimes (v \otimes w + w \otimes v - 2q(v, w)\,\mathrm{Id}) \otimes \chi : \kappa, \chi \in \bigotimes V\right\}.$$

The quotient is the Clifford algebra of (V, q),

$$\mathrm{Cl}(V, q) = \bigotimes V / \mathcal{I}_{\mathcal{cl}}(V),$$

it has an algebra structured induced by the tensor product in $\bigotimes V$ which satisfies

$$v \cdot w + w \cdot v = 2q(v, w)$$

whenever $v, w \in V$. By construction, $\mathrm{Cl}(V, q)$ has the universal property that any linear map $\mathcal{cl} : V \longrightarrow \mathrm{End}(W)$ satisfying (3.3) extends to a unique algebra homomorphism

$$\mathcal{cl} : \mathrm{Cl}(V, q) \longrightarrow \mathrm{End}(W), \text{ with } \mathcal{cl}\,(1) = \mathrm{Id}\,.$$

[2]Many authors use a different convention, namely $\mathcal{cl}\,(v)\,\mathcal{cl}\,(v) = -q(v)\,\mathrm{Id}\,.$

This universal property makes the Clifford algebra construction functorial. Indeed, if (V, q) and (V', q') are vector spaces with quadratic forms and $T : V \longrightarrow V'$ is a linear map such that $T^* q' = q$ then there is an induced algebra homomorphism

$$\widetilde{T} : \mathrm{Cl}(V, q) \longrightarrow \mathrm{Cl}(V', q').$$

Uniqueness of $\widetilde{T}$ shows that $\widetilde{T \circ S} = \widetilde{T} \circ \widetilde{S}$ for any quadratic-form preserving linear map S.

One advantage of functoriality is that we can extend this construction to vector bundles: Given a vector bundle $F \longrightarrow M$ and a bundle metric g_F there is a vector bundle

$$\mathrm{Cl}(F, g_F) \longrightarrow M$$

whose fiber over $p \in M$ is $\mathrm{Cl}(F_p, g_F(p))$.

In particular, coming back to our first order Dirac-type operator $D \in \mathcal{C}^\infty(M; E)$, we can conclude that its principal symbol $\sigma_1(D)$ induces an algebra bundle morphism

$$cl : \mathrm{Cl}(T^*M, g) \longrightarrow \mathrm{End}(E) \text{ with } cl(1) = \mathrm{Id}\,. \tag{3.5}$$

A bundle $E \longrightarrow M$ together with a map (3.5) is called a **Clifford module.** (It should really be called a bundle of Clifford modules, but the abbreviation is common.)

In the case of the de Rham operator we have

$$\sigma_1(d + \delta)(p, \xi) = -i(\mathfrak{e}_\xi - \mathfrak{i}_{\xi^\sharp}) : \Lambda^* T_p^* M \longrightarrow \Lambda^* T_p^* M,$$

and an induced algebra homomorphism

$$cl : \mathrm{Cl}(T^*M, g) \longrightarrow \mathrm{End}(\Lambda^* T_p^* M). \tag{3.6}$$

3.4 Clifford algebras

Let us consider the structure of the algebra $\mathrm{Cl}(V, q)$ associated to a quadratic form q on V. If $e_1, \ldots, e_k$ is a basis for V then any element of $\mathrm{Cl}(V, q)$ can be written as a polynomial in the e_i. In fact, using (3.4) we can choose the polynomial so that no exponent larger than one is used, and so that the e_i occur in increasing order; thus any $\omega \in \mathrm{Cl}(V, q)$ can be written in the form

$$\omega = \sum_{\alpha_j \in \{0,1\}} a_\alpha e_1^{\alpha_1} \cdot e_2^{\alpha_2} \cdots e_m^{\alpha_m} \tag{3.7}$$

with real coefficients a_α. This induces a filtration on $\mathrm{Cl}(V,q)$,

$$\mathrm{Cl}^{(k)}(V,q) = \{\omega \text{ as in (3.7), with } |\alpha| \leq k\},$$

which is consistent with the algebraic structure,

$$\mathrm{Cl}^{(k)}(V,q) \cdot \mathrm{Cl}^{(\ell)}(V,q) \subseteq \mathrm{Cl}^{(k+\ell)}(V,q).$$

The map (3.6), which there was induced by the symbol of the de Rham operator, always exists,

$$M : \mathrm{Cl}(V,q) \longrightarrow \mathrm{End}(\Lambda^* V).$$

There is a distinguished element $1 \in \Lambda^* V$, and evaluation at one induces

$$\widetilde{M} : \mathrm{Cl}(V,q) \longrightarrow \Lambda^* V, \quad \widetilde{M}(\omega) = M(\omega)(1).$$

Clearly if we think of $v \in V$ as an element of $\mathrm{Cl}(V,q)$, we have $\widetilde{M}(v) = v$. Moreover, comparing the anti-commutation rules in $\mathrm{Cl}(V,q)$ and $\Lambda^* V$, we can show that $\widetilde{M}$ is an isomorphism of vector spaces. In fact, $\Lambda^* V$ is the graded algebra associated to the filtered algebra $\mathrm{Cl}(V,q)$, i.e.,

$$\mathrm{Cl}^{(k)}(V,q)/\mathrm{Cl}^{(k-1)}(V,q) \cong \Lambda^k V.$$

If we set $q = 0$, then we recognize $\mathrm{Cl}(V,0) = \Lambda^* V$ as algebras. In general the algebras $\mathrm{Cl}(V,q)$ and $\Lambda^* V$ only coincide *as vector spaces.*

Let us identify a few of the low-dimensional Clifford algebras. From the vector space isomorphism $\mathrm{Cl}(V,q) \cong \Lambda^* V$ we know that $\dim \mathrm{Cl}(V,q)$ is a power of two. Let us agree that $\mathbb{R}$ itself, with its usual algebra structure, is the unique one-dimensional Clifford algebra.

In dimension two, let us identify $\mathrm{Cl}(\mathbb{R}, g_{\mathbb{R}})$ and $\mathrm{Cl}(\mathbb{R}, -g_{\mathbb{R}})$. Let e_1 be a unit vector in $\mathbb{R}$, so that both of these Clifford algebras are generated by 1, e_1. In the second case, where the quadratic form is minus the usual Euclidean norm, we have $e_1^2 = -1$, and it is easy to see that

$$\mathrm{Cl}(\mathbb{R}, -g_{\mathbb{R}}) \cong \mathbb{C}$$

as $\mathbb{R}$-algebras. In the first case, where the quadratic form is the usual Euclidean norm, the elements $v = \frac{1}{2}(1+e_1)$ and $w = \frac{1}{2}(1-e_1)$ satisfy

$$v \cdot w = 0, \quad v \cdot v = v, \quad w \cdot w = w$$

and we can identify

$$\mathrm{Cl}(\mathbb{R}, g_{\mathbb{R}}) \cong \mathbb{R} \oplus \mathbb{R}$$

as $\mathbb{R}$-algebras, where the summands consist of multiples of v and w, respectively.

Next consider $\mathrm{Cl}(\mathbb{R}^2, \pm g_{\mathbb{R}^2})$. Let e_1, e_2 be an orthonormal basis of $(\mathbb{R}^2, g_{\mathbb{R}^2})$. Using the universal property, we can define an isomorphism f between $\mathrm{Cl}(\mathbb{R}^2, \pm g_{\mathbb{R}^2})$ and a four dimensional algebra $\mathscr{A}$ by specifying the images of e_1 and e_2 and verifying that

$$f(e_i) \cdot f(e_i) = \pm 1, \quad i \in \{1, 2\}.$$

For $\mathrm{Cl}(\mathbb{R}^2, g_{\mathbb{R}^2})$ the map

$$e_1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

extends linearly to an algebra isomorphism

$$\mathrm{Cl}(\mathbb{R}^2, g_{\mathbb{R}^2}) \cong \mathrm{End}(\mathbb{R}^2).$$

Similarly, we can identify $\mathrm{Cl}(\mathbb{R}^2, -g_{\mathbb{R}^2})$ with the quaternions by extending the map

$$e_1 \mapsto i, \quad e_2 \mapsto j$$

(so, e.g., $e_1 \cdot e_2$ maps to k).

Thus (real) Clifford algebras generalize both the complex numbers and the quaternions. It is possible to compute all of the Clifford algebras of finite dimensional vector spaces[3] and an incredible fact is that they recur (up to 'suspension') with periodicity eight!

It is useful to consider the **complexified Clifford algebra**

$$\mathbb{C}l(V, q) = \mathbb{C} \otimes \mathrm{Cl}(V, q)$$

as these complex algebras have a simpler structure than $\mathrm{Cl}(V, q)$. In particular, we next show that they repeat with periodicity two.

Let us restrict attention to $V = \mathbb{R}^m$ and $q = g_{\mathbb{R}^m}$, and abbreviate

$$\mathrm{Cl}(\mathbb{R}^m, g_{\mathbb{R}^m}) = \mathrm{Cl}(m), \quad \mathbb{C}l(\mathbb{R}^m, g_{\mathbb{R}^m}) = \mathbb{C}l(m).$$

Proposition 8. *There are isomorphisms of complex algebras*

$$\mathbb{C}l(1) \cong \mathbb{C}^2, \quad \mathbb{C}l(2) \cong \mathrm{End}(\mathbb{C}^2)$$
$$\textit{and } \mathbb{C}l(m+2) \cong \mathbb{C}l(m) \otimes \mathbb{C}l(2)$$

and so

$$\mathbb{C}l(2k) \cong \mathrm{End}(\mathbb{C}^{2^k}), \quad \mathbb{C}l(2k+1) \cong \mathrm{End}(\mathbb{C}^{2^k}) \oplus \mathrm{End}(\mathbb{C}^{2^k}).$$

[3] See for instance, Lawson, H. Blaine, Jr. and Michelsohn, Marie-Louise Spin geometry. Princeton Mathematical Series, 38. Princeton University Press, Princeton, NJ, 1989.

Proof. The first two identities are just the complexifications of $\mathrm{Cl}(\mathbb{R}, g_{\mathbb{R}})$ and $\mathrm{Cl}(\mathbb{R}^2, g_{\mathbb{R}^2})$ computed above.

To prove periodicity, embed $\mathbb{R}^{m+2}$ into $\mathbb{C}l(m) \otimes \mathbb{C}l(2)$ by picking an orthonormal basis $\{e_1, \ldots, e_{m+2}\}$ and taking

$$e_j \mapsto i\,cl(e_j) \otimes cl(e_{m+1})cl(e_{m+2}) \text{ for } 1 \leq j \leq m,$$
$$e_j \mapsto 1 \otimes cl(e_j) \text{ for } j > m.$$

Then the universal property of $\mathbb{C}l(m+2)$ leads to the isomorphism above. □

Given a vector space with quadratic form (V, q), let $e_1, \ldots, e_m$ be an oriented orthonormal basis and define the **chirality operator**

$$\Gamma = i^{\lceil \frac{m}{2} \rceil} e_1 \cdots e_m \in \mathbb{C}l(V, q).$$

This is an involution (i.e., $\Gamma^2 = \mathrm{Id}$), and is independent of the choice of oriented orthonormal basis used in its definition. Moreover,

$$\Gamma \cdot v = -v \cdot \Gamma \text{ if } \dim V \text{ even}, \quad \Gamma \cdot v = v \cdot \Gamma \text{ if } \dim V \text{odd}.$$

From the explicit description of $\mathbb{C}l(m)$, we can draw some immediate consequences about the representations of this algebra[4]. Indeed if m is even, say $m = 2k$, then $\mathbb{C}l(2k) = \mathrm{End}(\mathbb{C}^{2^k})$ and up to isomorphism there is a unique irreducible $\mathbb{C}l(2k)$-module, $\not\Delta$, and it has dimension 2^k,

$$\mathbb{C}l(2k) = \mathrm{End}(\not\Delta).$$

If m is odd, say $m = 2k+1$, then $\mathbb{C}l(2k+1) = \mathrm{End}(\mathbb{C}^{2^k}) \oplus \mathrm{End}(\mathbb{C}^{2^k})$ and up to isomorphism there are two irreducible $\mathbb{C}l(2k+1)$ modules $\not\Delta_{\pm}$,

$$\mathbb{C}l(2k+1) = \mathrm{End}(\not\Delta_+) \otimes \mathrm{End}(\not\Delta_-)$$

each of dimension 2^k distinguished by

$$\Gamma\big|_{\not\Delta_{\pm}} = \pm 1.$$

Any finite dimensional complex representation of $\mathbb{C}l(2k)$ must be a direct sum of copies of $\not\Delta$, or equivalently of the form

$$W = \not\Delta \otimes W'$$

[4]For more details see, e.g., Roe, John Elliptic operators, topology and asymptotic methods. Second edition. Pitman Research Notes in Mathematics Series, 395. Longman, Harlow, 1998.

for some auxiliary 'twisting' vector space W'. We can recover W' from W via

$$W' = \mathrm{Hom}_{\mathbb{C}l(2k)}(\not\Delta, W)$$

and in the same fashion we have

$$\mathrm{End}(W) = \mathbb{C}l(2k) \otimes \mathrm{End}(W') = \mathbb{C}l(2k) \otimes \mathrm{End}_{\mathbb{C}l(2k)}(W).$$

One can similarly decompose representations of $\mathbb{C}l(2k+1)$ in terms $\not\Delta_{\pm}$.

Although $\mathbb{C}l(2k) = \mathrm{End}(\not\Delta)$ follows directly from the proposition, it can be useful to have a more concrete representative of $\not\Delta$. So let us describe how to construct $\not\Delta$ from $(\mathbb{R}^m, g_{\mathbb{R}^m})$ or indeed any m-dimensional vector space V with a positive-definite quadratic form, q. Let $e_1, \ldots, e_m$ be an oriented orthonormal basis of V and consider the subspaces of $V \otimes \mathbb{C}$,

$$K = \mathrm{span}\{\eta_j = \frac{1}{\sqrt{2}}(e_{2j-1} - ie_{2j}) : \quad j \in \{1, \ldots, m\}\}$$
$$\overline{K} = \mathrm{span}\{\overline{\eta}_j = \frac{1}{\sqrt{2}}(e_{2j-1} + ie_{2j}) : \quad j \in \{1, \ldots, m\}\}.$$

We can use the fact that $V \otimes \mathbb{C} = K \oplus \overline{K}$ to define a natural homomorphism

$$c\ell : V \otimes \mathbb{C} \longrightarrow \mathrm{End}(\Lambda^* K)$$
$$c\ell\,(k) = \sqrt{2}(\mathfrak{e}_k), \quad c\ell\left(\overline{k}\right) = \sqrt{2}(\mathfrak{i}_{\overline{k}}) \quad \text{for } k \in K.$$

This satisfies the Clifford relations and so extends to a homomorphism of $\mathbb{C}l(V, q)$ into $\mathrm{End}(\Lambda^* K)$. Comparing dimensions, we see this is in fact an isomorphism so we can take $\not\Delta = \Lambda^* K$. We refer to this representation as **the spin representation.** Below, we will use the consequence that the metric q on V induces a Hermitian metric on $\not\Delta$ compatible with Clifford multiplication.

3.5 Hermitian vector bundles

Recall that on a complex vector space V a Hermitian inner product is a map

$$\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{C}$$

that is linear in the first entry and satisfies

$$\langle v, w \rangle = \overline{\langle w, v \rangle}, \text{ for all } v, w \in V.$$

It is positive definite if $\langle v, v \rangle \geq 0$, and moreover $\langle v, v \rangle = 0$ implies $v = 0$.

A **Hermitian metric** h_E on a complex vector bundle $E \longrightarrow M$ is a smooth Hermitian inner product on the fibers of E. Compatible connections are defined in the usual way, namely, a connection ∇^E on E is compatible with the Hermitian metric h_E if

$$dh_E(\sigma, \tau) = h_E(\nabla^E \sigma, \tau) + h_E(\sigma, \nabla^E \tau), \text{ for all } \sigma, \tau \in \mathcal{C}^\infty(M; E).$$

One natural source of complex bundles is the complexification of real bundles. If $F \longrightarrow M$ is a real bundle of rank k then $F \otimes \mathbb{C}$ is a complex bundle of rank k. We can think of $F \otimes \mathbb{C}$ as $F \oplus F$ together with a bundle map

$$\begin{aligned} J : F \oplus F &\longrightarrow F \oplus F \\ (u, v) &\longmapsto (-v, u) \end{aligned}$$

satisfying $J^2 = -\operatorname{Id}$. (Such a map is called an *almost complex structure.*) A bundle metric g_F on $F \longrightarrow M$ induces a Hermitian metric h_F on $F \otimes \mathbb{C}$ by

$$h_F((u_1, v_1), (u_2, v_2)) = g_F(u_1, u_2) + g_F(v_1, v_2) + i\,(g_f(u_2, v_1) - g_F(u_1, v_2))$$

thus the 'real part' of h_F is g_F. Similarly a connection on F, ∇^F, induces a connection on $F \otimes \mathbb{C}$, namely the induced connection on $F \oplus F$. If ∇^F is compatible with g_F, then the induced connection on $F \otimes \mathbb{C}$ is compatible with h_F.

A Riemannian metric on a manifold induces a natural Hermitian inner product on $\mathcal{C}^\infty(M; \mathbb{C})$ or more generally on sections of a complex vector bundle E endowed with a Hermitian bundle metric, h_E. In the former case,

$$\langle f, h \rangle = \int_M f(p)\overline{h(p)} \ \mathrm{dvol}_g \ \text{ for all } f, h \in \mathcal{C}^\infty(M; \mathbb{C})$$

and in the latter,

$$\langle \sigma, \tau \rangle_E = \int_M h_E(\sigma, \tau) \ \mathrm{dvol}_g \ \text{ for all } \sigma, \tau \in \mathcal{C}^\infty(M; E)$$

(though the integrals need not converge if M is not compact).

A complex vector bundle E over a Riemannian manifold (M, g) is a **complex Clifford module** if there is a homomorphism, known as Clifford multiplication,

$$c\ell : \mathbb{C}l(T^*M, g) \longrightarrow \operatorname{End}(E).$$

Clifford multiplication is compatible with the Hermitian metric h_E if

$$h_E(c\ell\,(\theta)\,\sigma, \tau) = -h_E(\sigma, c\ell\,(\theta)\,\tau), \text{ for all } \theta \in \mathcal{C}^\infty(M; T^*M), \sigma, \tau \in \mathcal{C}^\infty(M; E).$$

Clifford multiplication is compatible with a connection ∇^E if

$$[\nabla^E, c\ell(\theta)] = c\ell(\nabla\theta),$$

where on the right hand side we are using the Levi-Civita connection. A connection compatible with Clifford multiplication is known as a **Clifford connection**.

Let us define a **Dirac bundle** to be a complex Clifford module, $E \longrightarrow M$, with a Hermitian metric h_E and a metric connection ∇^E, both of which are compatible with Clifford multiplication. Any Dirac bundle has a **generalized Dirac operator**

$$\eth_E \in \mathrm{Diff}^1(M; E)$$

given by

$$\eth_E : \mathcal{C}^\infty(M; E) \xrightarrow{\nabla^E} \mathcal{C}^\infty(M; T^*M \otimes E) \xrightarrow{ic\ell} \mathcal{C}^\infty(M; E).$$

Note that this is indeed a Dirac-type operator, since

$$[\eth_E, f] = [ic\ell \circ \nabla^E, f] = ic\ell(df)$$

shows that

$$\sigma_1(\eth_E)(p,\xi) = c\ell(\xi), \quad \sigma_2(\eth_E^2)(p,\xi) = |\xi|_g^2.$$

It is also a self-adjoint operator thanks to the compatibility of the metric with the connection and Clifford multiplication.

We know that $\eth_E^2$ is a Dirac-type operator and hence can be written as a connection Laplacian up to a zero-th order term. There is an explicit formula for this term which is often very useful.

First given two vector fields $V, W \in \mathcal{C}^\infty(M; TM)$, the curvature $R(V,W)$ can be thought of as a two-form

$$\widetilde{R}(V,W) \in \Omega^2(M), \quad \widetilde{R}(V,W)(X,Y) = g(R(V,W)Y, X) \text{ for all } X, Y \in \mathcal{C}^\infty(M; TM)$$

and so we have an action of $R(V,W)$ on E by $c\ell\left(\widetilde{R}(V,W)\right)$.

Next note that for any complex Clifford module, E, (for simplicity over an even dimensional manifold) and a contractible open set $\mathcal{U} \subseteq M$ we have

$$E\big|_{\mathcal{U}} \cong \underline{\not{\Delta}} \otimes \mathrm{Hom}_{\mathbb{C}l(T^*M)}(\underline{\not{\Delta}}, E\big|_{\mathcal{U}})$$

where $\underline{\not{\Delta}}$ denotes the trivial bundle $\not{\Delta} \times \mathcal{U} \longrightarrow \mathcal{U}$. Correspondingly we have

$$\mathrm{End}(E) = \mathbb{C}l(T^*M) \otimes \mathrm{End}_{\mathbb{C}l(T^*M)}(E)$$

locally. However both sides of this equality define bundles over M and the natural map from the right hand side to the left hand side is locally an isomorphism so these are in fact globally isomorphic bundles.

Proposition 9. *The curvature* $R^E \in \Omega^2(M; \operatorname{End}(E))$ *of a Clifford connection* ∇^E *on* $E \longrightarrow M$ *decomposes under the isomorphism* $\operatorname{End}(E) = \mathbb{C}l(T^*M) \otimes \operatorname{End}_{\mathbb{C}l(T^*M)}(E)$ *into a sum*

$$R^E(V,W) = \tfrac{1}{4}c\!\ell\left(\widetilde{R}(V,W)\right) + \widehat{R}^E(V,W),$$
$$c\!\ell\left(\widetilde{R}(V,W)\right) \in \mathbb{C}l(T^*M), \quad \widehat{R}^E(V,W) \in \operatorname{End}_{\mathbb{C}l(T^*M)}(E)$$

for any $V, W \in \mathcal{C}^\infty(M; TM)$.

We call $\widehat{R}^E \in \Omega^2(M; \operatorname{End}_{\mathbb{C}l(T^*M)}(E))$ the **twisting curvature** of the Clifford connection ∇^E.

Proof. From the compatibility of the connection with Clifford multiplication, and the definition of the curvature in terms of the connection, it is easy to see that for any $V, W \in \mathcal{C}^\infty(M; TM)$, $\omega \in \mathcal{C}^\infty(M; T^*M)$, and $s \in \mathcal{C}^\infty(M; E)$,

$$R^E(V,W)c\!\ell\,(\omega)\, s = c\!\ell\,(\omega)\, R^E(V,W)s + c\!\ell\,(R(V,W)\omega)\, s$$

where $R(V,W)\omega$ denotes the action of $R(V,W)$ as a map from one-forms to one-forms. We can write the last term as a commutator

$$c\!\ell\,(R(V,W)\omega)\, s = \tfrac{1}{4}c\!\ell\left(\widetilde{R}(V,W)\right) c\!\ell\,(\omega)\, s - \tfrac{1}{4}c\!\ell\,(\omega)\, c\!\ell\left(\widetilde{R}(V,W)\right) s$$

since we have, in any local coordinates θ^i for T^*M with dual coordinates ∂_i,

$$\begin{aligned} &c\!\ell\left(\widetilde{R}(V,W)\right) c\!\ell\,(\theta^p) - c\!\ell\,(\theta^p)\, c\!\ell\left(\widetilde{R}(V,W)\right) \\ &= g(R(V,W)\partial_i, \partial_j)c\!\ell\left(\theta^i \cdot \theta^j \cdot \theta^p\right) - g(R(V,W)\partial_i, \partial_j)c\!\ell\left(\theta^p \cdot \theta^i \cdot \theta^j\right) \\ &= -2g(R(V,W)\partial_p, \partial_j)c\!\ell\left(\theta^j\right) + 2g(R(V,W)\partial_i, \partial_p)c\!\ell\left(\theta^i\right) \\ &= 4g(R(V,W)\partial_i, \partial_p)c\!\ell\left(\theta^i\right). \end{aligned}$$

So we conclude that

$$R^E - \tfrac{1}{4}c\!\ell\,(R)$$

commutes with Clifford multiplication by one-forms, and hence it commutes with all Clifford multiplication. $\square$

With this preliminary out of the way we can now compare $\eth_E^2$ and Δ^∇. The proof is through a direct computation in local coordinates[5].

[5]See Theorem 3.52 of NICOLE BERLINE, EZRA GETZLER, MICHÈLE VERGNE Heat kernels and Dirac operators. Corrected reprint of the 1992 original. Grundlehren Text Editions. Springer-Verlag, Berlin, 2004. x+363 pp. ISBN: 3-540-20062-2

Theorem 10 (Lichnerowicz formula)**.** *Let $E \longrightarrow M$ be a Dirac bundle, Δ^∇ the Bochner Laplacian of the Clifford connection ∇^E, $R^{E'} \in \Omega^2(M; \mathrm{End}_{\mathbb{C}l(T^*M)}(E))$ the twisting curvature of ∇^E and scal_M the scalar curvature of M, then*

$$\eth_E^2 = \Delta^\nabla + cl\left(\widehat{R}^E\right) + \tfrac{1}{4}\mathrm{scal}_M$$

where we define (in local coordinates ∂_i for TM and dual coordinates θ^i)

$$cl\left(\widehat{R}^{E'}\right) = \sum_{i<j} \widehat{R}^E(\partial_i, \partial_j) cl\left(\theta^i \cdot \theta^j\right).$$

A $\mathbb{Z}_2$**-graded Dirac bundle** is a Dirac bundle E with a decomposition

$$E = E^+ \oplus E^-$$

that is orthogonal with respect to the metric, parallel with respect to the connection, and odd with respect to the Clifford multiplication – by which we mean that

$$cl\,(v) \text{ interchanges } E^+ \text{ and } E^-, \text{ for all } v \in T^*M.$$

We can equivalently express this in terms of an involution on E : A $\mathbb{Z}_2$ grading on a Dirac bundle is

$$\gamma \in \mathcal{C}^\infty(M; \mathrm{End}(E))$$

satisfying

$$\gamma^2 = \mathrm{Id}, \quad \gamma cl\,(v) = -cl\,(v)\,\gamma, \quad \nabla^E\gamma = 0, \quad \gamma^* = \gamma.$$

We make E into a $\mathbb{Z}_2$-graded Dirac-bundle by defining

$$E^\pm = \{v \in E : \gamma(v) = \pm v\}.$$

If M is even dimensional, then any complex Clifford module over M has a $\mathbb{Z}_2$-grading given by Clifford multiplication by the chirality operator

$$\gamma = cl\,(\Gamma)$$

but there are often other $\mathbb{Z}_2$-gradings as well. (Note that Γ is a well-defined section of $\mathbb{C}l(T^*M)$ by naturality.)

In the presence of a $\mathbb{Z}_2$ grading, the generalized Dirac operator is a graded (or odd) operator,

$$\eth_E = \begin{pmatrix} 0 & \eth_E^- \\ \eth_E^+ & 0 \end{pmatrix}, \quad \text{with } \eth_E^\pm \in \mathrm{Diff}^1(M; E^\pm, E^\mp), \quad (\eth_E^+)^* = \eth_E^-.$$

The index of the Dirac-type operator $\eth_E^+$ is the difference

$$\operatorname{ind} \eth_E^+ = \dim \ker \eth_E^+ - \dim \ker \eth_E^-.$$

and often this is an interesting geometric, or even topological, quantity. Generalized Dirac operators show up regularly in geometry, we will describe several natural examples in the next section.

A simple consequence of the Lichnerowicz formula is that

$$c\ell\left(\widehat{R}^E\right) + \tfrac{1}{4}\mathrm{scal}_M > 0 \implies \ker \eth_E = \{0\} \implies \operatorname{ind} \eth_E^+ = 0.$$

So the vanishing of the index of $\eth_E^+$ is an obstruction to having $c\ell\left(\widehat{R}^E\right) + \frac{1}{4}\mathrm{scal}_M > 0$. This is most interesting when the former is a topological invariant.

3.6 Examples

3.6.1 The Gauss-Bonnet operator

Our usual example is the de Rham operator

$$\eth_{\mathrm{dR}} = d + \delta : \Omega^*(M) \longrightarrow \Omega^*(M),$$

although this is a *real* generalized Dirac operator, so our conventions will have to be slightly different from those above. As we mentioned above, in this case the Clifford module is the bundle of forms of all degrees,

$$\Lambda^* T^* M \longrightarrow M.$$

Clifford multiplication by a 1-form $\omega \in \mathcal{C}^\infty(M; T^*M)$ is given by

$$c\ell\,(\omega) = (\mathfrak{e}_\omega - \mathfrak{i}_{\omega^\sharp}),$$

this is compatible with both the Riemannian metric on $\Lambda^* T^* M$ and the Levi-Civita connection, and we have

$$c\ell \circ \nabla = d + \delta.$$

Thus $\Lambda^* T^* M$ is a (real) Dirac bundle with generalized Dirac operator the de Rham operator.

This Dirac bundle is $\mathbb{Z}_2$-graded. Let us define

$$\Lambda^{\mathrm{even}} T^* M = \bigoplus_{k\,\mathrm{even}} \Lambda^k T^* M, \quad \Lambda^{\mathrm{odd}} T^* M = \bigoplus_{k\mathrm{odd}} \Lambda^k T^* M$$

and correspondingly $\Omega^{\text{even}}(M)$ and $\Omega^{\text{odd}}(M)$. Notice that Clifford multiplication exchanges these subbundles and that the covariant derivative with respect to a vector field preserves these subbundles, so

$$\eth_{\text{dR}}^{\text{even}} : \Omega^{\text{even}}(M) \longrightarrow \Omega^{\text{odd}}(M), \quad \eth_{\text{dR}}^{\text{odd}} : \Omega^{\text{odd}}(M) \longrightarrow \Omega^{\text{even}}(M).$$

The operator $\eth_{\text{dR}}^{\text{even}}$ is the **Gauss-Bonnet operator**.

A form in the null space of $d + \delta$ is necessarily in the null space of the Hodge Laplacian $\Delta = (d + \delta)^2$ and so is a harmonic form. Conversely any compactly supported harmonic form is in the null space of $d + \delta$. So on compact manifolds,

$$\ker(d + \delta) = \mathcal{H}^*(M), \quad \operatorname{ind} \eth_{\text{dR}}^{\text{even}} = \dim \mathcal{H}^{\text{even}}(M) - \dim \mathcal{H}^{\text{odd}}(M).$$

The latter is the *Euler characteristic of the Hodge cohomology of* M,

$$\chi_{\text{Hodge}}(M),$$

an important *topological* invariant of M. Notice that on odd-dimensional compact manifolds, the Hodge star is an isomorphism between $\mathcal{H}^{\text{even}}(M)$ and $\mathcal{H}^{\text{odd}}(M)$, so the Euler characteristic of the Hodge cohomology vanishes.

The Lichnerowicz formula in this case is known as the Weitzenböck formula, and simplifies to

$$\Delta_k = \nabla^* \nabla - \sum_{ijk\ell} g(R(e_i, e_j)e_k, e_\ell) \mathfrak{e}_{\theta^k} \mathfrak{i}_{e_\ell} \mathfrak{e}_{\theta^i} \mathfrak{i}_{\theta^j}$$

where e_k is a local orthonormal frame with dual coframe θ^k. In particular on functions we get

$$\Delta_0 = \nabla^* \nabla$$

and on one-forms we get *Bochner's formula*

$$\Delta_1 = \nabla^* \nabla + \widetilde{\text{Ric}}$$

where $\widetilde{\text{Ric}}$ is the endomorphism of T^*M associated to the Ricci curvature via the metric. In particular, if the Ricci curvature is bounded below by a positive constant then Δ_1 does not have null space, i.e.,

$$\text{Ric} > 0 \implies \mathcal{H}^1(M) = 0.$$

3.6.2 The signature operator

Now consider the complexification of the differential forms on M,

$$\Lambda^*_{\mathbb{C}} T^*M = \Lambda^* T^*M \otimes \mathbb{C}$$

and the corresponding space of sections, $\Omega^*_{\mathbb{C}}(M)$. Clifford multiplication is as before, but now we can allow complex coefficients,

$$c\ell\,(\omega) = \tfrac{1}{i}(\mathfrak{e}_\omega - \mathfrak{i}_{\omega^\sharp}),$$

this is compatible with both the Riemannian metric on $\Lambda^*_{\mathbb{C}} T^*M$ (extended to a Hermitian metric) and the Levi-Civita connection, and we have

$$i c\ell \circ \nabla = d + \delta.$$

Since $\Lambda^*_{\mathbb{C}} T^*M$ is a complex Clifford module, we have an involution given by Clifford multiplication by the chirality operator Γ. One can check that, up to a factor of i, Γ coincides with the Hodge star,

$$c\ell\,(\Gamma) : \Omega^k_{\mathbb{C}}(M) \longrightarrow \Omega^{m-k}_{\mathbb{C}}(M), \quad c\ell\,(\Gamma) = i^{k(k-1)+\frac{1}{2}\dim M} * .$$

If M is even dimensional, this induces a $\mathbb{Z}_2$ grading of $\Lambda^*_{\mathbb{C}} T^*M$ as a Dirac bundle,

$$\Lambda^*_{\mathbb{C}} T^*M = \Lambda^*_+ T^*M \oplus \Lambda^*_- T^*M.$$

The de Rham operator splits as

$$d + \delta = \begin{pmatrix} 0 & \eth^-_{\text{sign}} \\ \eth^+_{\text{sign}} & 0 \end{pmatrix}$$

and we refer to $\eth^+_{\text{sign}}$ as the **Hirzebruch signature operator**.

The index of this operator

$$\operatorname{ind} \eth^+_{\text{sign}} = \dim\ker \eth^+_{\text{sign}} - \dim\ker \eth^-_{\text{sign}}$$

is again an important topological invariant of M. It is always zero if $\dim M$ is not a multiple of four, and otherwise it is equal to the signature of the quadratic form

$$\begin{aligned} \mathcal{H}^{\frac{1}{2}\dim M}(M) &\longrightarrow \mathbb{C} \\ u &\longmapsto \int_M u \wedge u \end{aligned}$$

It is called the *signature of* M, $\operatorname{sign}(M)$.

3.6.3 The Dirac operator

If $E \longrightarrow M$ is a complex Clifford module over manifold M of even dimension, say $m = 2k$, then Clifford multiplication

$$\mathcal{cl} : \mathbb{C}l(T^*M, g) \longrightarrow \mathrm{End}(E)$$

is, for each $p \in M$, a finite dimensional complex representation of $\mathbb{C}l(2k)$ on E_p. We can decompose E_p in terms of the unique irreducible representation of $\mathbb{C}l(T_p^*M)$, $\not\Delta$,

$$E_p = \not\Delta \otimes \mathrm{Hom}_{\mathbb{C}l(2k)}(\not\Delta, E_p).$$

If it happens that $E_p = \not\Delta$ as Clifford modules for all $p \in M$ we say that E is a **spin$^{\mathbb{C}}$ bundle** over M. If a spin$^{\mathbb{C}}$ bundle satisfies the symmetry condition

$$\mathrm{Hom}_{\mathbb{C}l(T^*M)}(\overline{E}, E) \text{ is trivial}$$

we say that E is a **spin bundle** over M. We will generally denote a spin bundle over M by

$$\not S \longrightarrow M.$$

Remark 5. In the interest of time we have avoided talking about *principal bundles* and *structure groups* and hence also *spin structures* and *spin$^{\mathbb{C}}$ structures*. In brief, the structure group of a $\mathbb{R}^k$-vector bundle is $GL_k(\mathbb{R})$ because the transition maps between trivializations are valued in $GL_k(\mathbb{R})$. If we choose a metric on the bundle then we can restrict attention to trivializations where the transition maps are valued in $O(n)$, and if we choose an orientation we can use oriented frames and reduce the structure group to $SO(n)$. It turns out that a double cover of $SO(n)$ is a Lie group called the *spin group* $Spin(n)$. (We can identify $Spin(n)$ with the multiplicative group of invertible elements in $\mathrm{Cl}^{(\mathrm{even})}(\mathbb{R}^n, g_{\mathbb{R}^n})$.) If the structure group of the tangent bundle of a manifold M can be 'lifted' to $Spin(n)$ we say that M admits a spin structure, and we can combine a spin structure with the spin representation to obtain a spin bundle. The group $Spin^{\mathbb{C}}(n)$ is the subgroup of $\mathbb{C}l(\mathbb{R}^k, g_{\mathbb{R}^k})$ generated by $Spin(n)$ together with the unit complex numbers. If the structure group of the tangent bundle of a manifold M can be 'lifted' to $Spin^{\mathbb{C}}(n)$ we say that M admits a spin$^{\mathbb{C}}$ structure, and we can combine a spin$^{\mathbb{C}}$ structure with the spin representation to obtain a spin$^{\mathbb{C}}$ bundle. For details we refer to any of the books in the bibliography of this lecture.

There are topological obstructions to the existence of a spin bundle over a manifold M and the number of spin bundles when they do exist is also determined topologically[6]. For example, a genus g Riemann surface admits 2^{2g} distinct spin

[6] For a thorough discussion see LAWSON, H. BLAINE, JR. AND MICHELSOHN, MARIE-LOUISE Spin geometry. Princeton Mathematical Series, 38. Princeton University Press, Princeton, NJ, 1989.

bundles. All spheres in dimension greater than two admit spin bundles. All compact orientable manifolds of dimension three or less admit spin bundles. The complex projective space $\mathbb{CP}^k$ admits a spin bundle if and only if k is odd. Additionally, any (almost) complex manifold has a $\text{spin}^{\mathbb{C}}$ bundle.

The Riemannian metric and Levi-Civita connection induce the structure of a Dirac bundle on any spin bundle. Indeed, we saw above that the metric on a $2k$-dimensional vector space V with positive quadratic form induces a metric on the corresponding $\not{\Delta}$ so it suffices to describe the induced compatible connection. In a local coordinate chart $\mathcal{U}$ that trivializes T^*M, if we choose an orthonormal frame $\theta^1, \ldots, \theta^m$ for $TM\big|_{\mathcal{U}}$, the Levi-Civita connection is determined by an $m \times m$ matrix of one-forms ω defined through

$$\nabla \theta^i = \omega_{ij} \theta^j.$$

(The fact that the connection respects the metric is reflected in $\omega_{ij} = -\omega_{ji}$.) We can use ω to define a one-form valued in $\text{End}(\not{S})\big|_{\mathcal{U}}$, namely

$$\widetilde{\omega} = \frac{1}{2} \sum_{i<j} \omega_{ij} \, c\ell \left(\theta^i \cdot \theta^j \right)$$

and the induced connection on $\not{S}$ over $\mathcal{U}$ is given by $d + \widetilde{\omega}$. One can check that this local description does define a connection on $\not{S}$ which then must be metric and Clifford.

The resulting generalized Dirac operator is called *the* Dirac operator (associated to the spin bundle $\not{S}$)

$$\eth : \mathcal{C}^\infty(M; \not{S}) \longrightarrow \mathcal{C}^\infty(M; \not{S}).$$

As before Clifford multiplication by the chirality operator Γ defines an involution exhibiting a $\mathbb{Z}_2$-grading on $\not{S}$,

$$\not{S} = \not{S}^+ \oplus \not{S}^-,$$

and a corresponding splitting of the Dirac operator

$$\eth = \begin{pmatrix} 0 & \eth^- \\ \eth^+ & 0 \end{pmatrix}.$$

The Lichnerowicz formula for the Dirac operator is particularly interesting since there is no twisting curvature, so we get

$$\eth^2 = \Delta^\nabla + \frac{1}{4} \text{scal}_M.$$

Thus if the metric has positive scalar curvature then $\eth^2$ is strictly positive and in particular does not have null space. We will show that the index of $\eth^+$ is a topological invariant. Thus the vanishing of $\operatorname{ind}\eth^+$ is a topological necessary condition for the existence of a metric of positive scalar curvature[7] on M.

3.6.4 Twisted Clifford modules

If $E \longrightarrow M$ is a Dirac bundle and $F \longrightarrow M$ is any Hermitian bundle with a unitary connection, then $E \otimes F \longrightarrow M$ is again a Dirac bundle if we take the tensor product metric and connection and extend Clifford multiplication to $E \otimes F$ by acting on the first factor. We say that this is the Dirac bundle E **twisted by** F.

If M admits a spin bundle $\$ \longrightarrow M$, then any Dirac bundle $E \longrightarrow M$ is a twist of $\$$. Indeed, we have

$$E = \$ \otimes \mathrm{Hom}_{\mathbb{C}l(T^*M)}(\$, E).$$

In the case of a twisted spin bundle $\$ \otimes F \longrightarrow M$, the 'twisting curvature' in the Lichnerowicz formula is precisely the curvature of the connection on F.

The index of a twist of the signature operator is known as a **twisted signature.** Twisted Dirac and twisted signature operators play an important rôle in the heat equation proof of the index theorem.

For example if M admits a spin bundle $\$ \longrightarrow M$ then the Clifford module $\Lambda^* T^*M \otimes \mathbb{C}$ must be a twist of $\$$, and indeed

$$\Lambda^*_{\mathbb{C}} T^*M \cong \mathbb{C}l(T^*M) \cong \mathrm{End}(\$) = \$^* \otimes \$.$$

The generalized Dirac operator $d + \delta$ on $\Lambda^*_{\mathbb{C}} T^*M$ is the twisted Dirac operator obtained by twisting $\$$ with $\$^*$. The two different $\mathbb{Z}_2$ gradings correspond to whether we treat $\$^*$ as a $\mathbb{Z}_2$-graded bundle (which yields the splitting into $\Lambda^{\mathrm{even}}_{\mathbb{C}} T^*M$ and $\Lambda^{\mathrm{odd}}_{\mathbb{C}} T^*M$) or as an ungraded bundle (which yields the splitting into $\Lambda^*_+ T^*M$ and $\Lambda^*_- T^*M$).

3.6.5 The $\overline{\partial}$ operator

A real manifold M of dimension $m = 2k$ is a **complex manifold** if it has a collection of local charts parametrized by open sets in $\mathbb{C}^k$ for which the transition charts are

[7] In contrast, any manifold of dimension at least three admits a metric of constant negative scalar curvature, see Aubin, Thierry *Métriques riemanniennes et courbure.* J. Differential Geometry **4**, 1970, 383-424. The same is true for manifolds with boundary and for non-compact manifolds. For compact surfaces, the sign of the integral of the scalar curvature is topologically determined by the Gauss-Bonnet theorem.

holomorphic. This yields a decomposition of the complexified tangent bundle

$$TM \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M, \quad v \in T_p^{(1,0)}M \iff \overline{v} \in T_p^{0,1}M.$$

By duality there is a similar decomposition of the cotangent bundle

$$T^*M \otimes \mathbb{C} = \Lambda^{1,0}M \oplus \Lambda^{01,}M.$$

In local complex coordinates, $z_1, \dots, z_m$, which we can decompose into real and imaginary parts $z_j = x_j + iy_j$. We can define

$$dz_j = dx_j + idy_j, \quad d\overline{z}_j = dx_j - idy_j$$

and then $\Lambda^{1,0}M$ is spanned by $\{dz_j\}$ and $\Lambda^{0,1}M$ is spanned by $\{d\overline{z}_j\}$.

More generally we set

$$\Lambda^{p,q}M = \Lambda^p(\Lambda^{1,0}M) \wedge \Lambda^q(\Lambda^{0,1}M)$$

and we have a natural isomorphism

$$\Lambda^k T^*M \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q}M.$$

We denote the space of sections of $\Lambda^{p,q}M$ by $\Omega^{p,q}M$, and speak of forms of bidegree (p,q).

The exterior derivative naturally maps

$$d : \Omega^{p,q}M \longrightarrow \Omega^{p+1,q}M \oplus \Omega^{p,q+1}M$$

and we can decompose it into the **Dolbeault operators**

$$d = \partial + \overline{\partial}$$
$$\partial : \Omega^{p,q}M \longrightarrow \Omega^{p+1,q}M, \quad \overline{\partial} : \Omega^{p,q}M \longrightarrow \Omega^{p,q+1}M.$$

It follows from $d^2 = 0$ that

$$\partial^2 = \overline{\partial}^2 = \partial\overline{\partial} + \overline{\partial}\partial = 0$$

and from the Leibnitz rule for d that

$$\partial(\omega \wedge \eta) = \partial\omega \wedge \eta + (-1)^{|\omega|}\omega \wedge \partial\eta$$
$$\overline{\partial}(\omega \wedge \eta) = \overline{\partial}\omega \wedge \eta + (-1)^{|\omega|}\omega \wedge \overline{\partial}\eta.$$

In local coordinates, for $\omega \in \Omega^{p,q}M$

$$\omega = \sum_{|J|=p,|K|=q} \alpha_{JK} dz^J \wedge d\overline{z}^K \implies$$

$$\partial\omega = \sum_{|J|=p,|K|=q,\ell} \frac{\alpha_{JK}}{\partial z_\ell} dz^\ell \wedge dz^J \wedge d\overline{z}^K, \text{ and } \overline{\partial}\omega = \sum_{|J|=p,|K|=q,\ell} \frac{\alpha_{JK}}{\partial \overline{z}_\ell} d\overline{z}^\ell \wedge dz^J \wedge d\overline{z}^K.$$

The Dolbeault complex is given by

$$0 \longrightarrow \Omega^{0,0}M \xrightarrow{\overline{\partial}} \Omega^{0,1}M \xrightarrow{\overline{\partial}} \dots \xrightarrow{\overline{\partial}} \Omega^{0,m}M \longrightarrow 0 \tag{3.8}$$

and its cohomology is called the Dolbeault cohomology of M. We define the Hodge-Dolbeault cohomology to be

$$\mathcal{H}^{0,q}(M) = \{\omega \in \Omega^{0,p}M : (\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial})\omega = 0\}.$$

As in the real case, we will show below that the Hodge-Dolbeault cohomology coincides with the Dolbeault cohomology.

A complex Hermitian manifold M is a **Kähler manifold** if the splitting

$$TM \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$$

is preserved by the Levi-Civita connection. On a Kähler manifold, the bundle

$$\Lambda^{0,*}M = \bigoplus_k \Lambda^{0,k}M$$

is a Dirac bundle with respect to the Hermitian metric on M, the Levi-Civita connection, and the Clifford multiplication

$$c\ell\,(\omega) = \sqrt{2}\,(\mathfrak{e}_{\omega^{0,1}} - \mathfrak{i}_{(\omega^{1,0})^\sharp}).$$

The corresponding generalized Dirac operator is

$$\sqrt{2}\,(\overline{\partial} + \overline{\partial}^*).$$

There is a natural $\mathbb{Z}_2$ grading on $\Lambda^{0,*}M$ given by

$$\Lambda^{0,*}M = \Lambda^{0,\mathrm{even}}M \oplus \Lambda^{0,\mathrm{odd}}M.$$

The index of the corresponding non-self-adjoint operator is the Euler characteristic of the Hodge-Dolbeault cohomology.

3.7 Exercises

Exercise 1.
a) Show that the scalar linear differential operators on $\mathbb{R}^m$ form a (non-commutative) algebra. Verify that the commutator $PQ - QP$ of a differential operator of order ℓ and a differential operator of order ℓ' has at most order $\ell + \ell' - 1$.
b) Show that the linear differential operators with constant coefficients form a commutative subalgebra which is isomorphic to the polynomial algebra $\mathbb{R}[\xi_1, \dots, \xi_m]$.

Exercise 2.
Verify that for $P \in \mathrm{Diff}^k(M; E, F)$ and $Q \in \mathrm{Diff}^\ell(M; F, G)$, the operator PQ is in $\mathrm{Diff}^{k+\ell}(M; E, G)$ and its principal symbol satisfies $\sigma_{k+\ell}(P \circ Q) = \sigma_k(P) \circ \sigma_\ell(Q)$.

Exercise 3.
Let $E \longrightarrow M$ and $F \longrightarrow M$ be bundles endowed with metrics, lying over the Riemannian manifold M. Verify that the formal adjoint of $P \in \mathrm{Diff}^k(M; E, F)$, P^*, is an element of $\mathrm{Diff}^k(M; F, E)$ and its principal symbol satisfies $\sigma_k(P^*) = \sigma_k(P)^*$.

Exercise 4.
Let M be a Riemannian manifold with boundary, and $E \longrightarrow M$, and $F \longrightarrow M$ bundles endowed with bundle metrics. The formal adjoint of an operator $P : \mathcal{C}^\infty(M; E) \longrightarrow \mathcal{C}^\infty(M; F)$ is defined by requiring

$$\langle P\sigma, \tau\rangle_F = \langle \sigma, P^*\tau\rangle_E, \text{ for all } \sigma \in \mathcal{C}_c^\infty(M^\circ; E), \tau \in \mathcal{C}_c^\infty(M^\circ; F).$$

Let ν be the inward-pointing unit normal vector field at the boundary. Define $\mathrm{dvol}_{\partial M}$ by $\mathrm{i}_\nu\, \mathrm{dvol}_M \big|_{\partial M}$. Show that if $P \in \mathrm{Diff}^1(M; E, F)$ is a *first-order* differential operator then

$$\langle P\sigma, \tau\rangle_F - \langle \sigma, P^*\tau\rangle_E = \int_{\partial M} g_F(i\sigma_1(P)(p, \nu^\flat)\sigma, \tau)\ \mathrm{dvol}_{\partial M},$$
$$\text{for all } \sigma \in \mathcal{C}_c^\infty(M; E), \tau \in \mathcal{C}_c^\infty(M; F).$$

Hint: Use a partition of unity to reduce to a coordinate chart, then compute in local coordinates.

Exercise 5.
Show that $\mathrm{End}(\mathbb{C}^m)$ has no proper two-sided ideals.
Hint: Suppose M_0 belongs to such an ideal $\mathcal{I}$ and $v_0 \neq 0$ belongs to the range of M_0. Show that every $v \in \mathbb{C}^m$ belongs to the range of some $M \in \mathcal{I}$, and hence that every one-dimensional projection belongs to $\mathcal{I}$.

Exercise 6.
a) Suppose a compact oriented manifold M has non-negative Ricci curvature – i.e., that $g(\widetilde{\mathrm{Ric}}(\omega), \omega) \geq 0$ for all $\omega \in \Omega^1(M)$. Show that every harmonic one-form η satisfies $\nabla\eta = 0$. As this is a first order differential equation for η, a harmonic one-form

is determined by its 'initial condition' at a point in M. Conclude that, if M is also connected, $\dim \mathcal{H}^1(M) \leq \dim M$.
b) Show that on any connected compact oriented Riemannian manifold $\dim \mathcal{H}^0(M) = 1$ and so by Poincaré duality $\dim \mathcal{H}^m(M) = 1$.
c) From (b), we see that on a compact oriented Riemannian surface (i.e., two dimensional manifold) $\chi_{\mathrm{Hodge}}(M) = 2 - \dim \mathcal{H}^1(M)$. Show that if M is a compact oriented surface admitting a metric of non-negative Gaussian curvature then $\chi_{\mathrm{Hodge}}(M) \geq 0$. (It turns out that only the sphere and the torus have non-negative Euler characteristic.)

Exercise 7.
Let M and M' be compact oriented Riemannian manifolds.
i)Show that

$$\dim \mathcal{H}^k(M \times M') = \sum_{j=0}^{k} (\dim \mathcal{H}^j(M))(\dim \mathcal{H}^{k-j}(M'))$$

when $M \times M'$ is endowed with the product metric (we will soon show that $\dim \mathcal{H}^k(M)$ is independent of the choice of metric).
ii) Conclude that

$$\chi_{\mathrm{Hodge}}(M \times M') = \chi_{\mathrm{Hodge}}(M)\chi_{\mathrm{Hodge}}(M').$$

Hint: Use the fact that $\Lambda^k T^*(M \times M') = \bigoplus_{j=0}^k (\Lambda^j T^* M) \wedge (\Lambda^{k-j} T^* M')$, show that with respect to this decomposition the Hodge Laplacian for the product metric satisfies

$$\Delta_{k,M\times M'} = \sum_{j=0}^{k} (\Delta_{j,M} \otimes \mathrm{Id} + \mathrm{Id} \otimes \Delta_{k-j,M'}),$$

and conclude that if

$$\{\alpha_1^j, \ldots, \alpha_{b_j(M)}^j\} \text{ is a basis of } \mathcal{H}^j(M) \text{ and } \{\beta_1^j, \ldots, \beta_{b_j(M')}^j\} \text{ is a basis of } \mathcal{H}^j(M')$$

then $\{\alpha_s^j \otimes \beta_t^{k-j} : 0 \leq j \leq k\}$ is a basis of $\mathcal{H}^k(M \times M')$.

Exercise 8.
Let M and M' be compact oriented Riemannian manifolds.
Show that signature is multiplicative,

$$\mathrm{sign}(M \times M') = \mathrm{sign}(M)\mathrm{sign}(M'),$$

where the signature on $M \times M'$ is defined using the product metric (we will soon show that the signature is independent of the choice of metric).
Hint: Express $d_{M\times M'}$ and $\delta_{M\times M'}$ in terms of d_M, δ_M, $d_{M'}$, and $\delta_{M'}$. Choose bases of $\Lambda_\pm T^*M$ and $\Lambda_\pm T^*M'$ and use them to construct bases of $\Lambda_\pm T^*(M \times M')$.

Exercise 9.
We can interpret any vector field V on M as a first order differential operator. What is its principal symbol?

Exercise 10.
Let $E \longrightarrow M$ and $F \longrightarrow M$ be two vector bundles over M. Show that, for any $k \in \mathbb{N}$, the assignment $\mathcal{U} \mapsto \mathrm{Diff}^k(M; E\big|_{\mathcal{U}}, F\big|_{\mathcal{U}})$ is a sheaf on M.

3.8 Bibliography

The books *Spin geometry* by Lawson and Michelson, and *Heat kernels and Dirac operators* by Berline, Getzler, and Vergne treat Dirac operators in detail. Other good sources are *Elliptic operators, topology and asymptotic methods* by Roe which gives a very nice short treatment of many of the topics of this course, and volume two of *Partial differential equations* by Taylor.

There is also a very nice treatment of these topics in Melrose, *The Atiyah-Patodi-Singer index theorem,* where they are systematically extended to non-compact manifolds with asymptotically cylindrical ends.

Lecture 4

Pseudodifferential operators on $\mathbb{R}^m$

4.1 Distributions

It is an unfortunate fact that most functions are not differentiable. Happily, there is a way around this through duality. For instance, if $f \in \mathcal{C}^\infty(\mathbb{R})$[1] and $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ then

$$\int_{\mathbb{R}} f'(t)\phi(t)\ dt = -\int_{\mathbb{R}} f(t)\phi'(t)\ dt$$

and the right hand side makes sense even if f is not differentiable. So for a possibly non-differentiable f we will say that a function h is a **weak derivative of** f if

$$\int_{\mathbb{R}} f(t)\phi'(t)\ dt = -\int_{\mathbb{R}} h(t)\phi(t)\ dt, \qquad \text{for all } \phi \in \mathcal{C}_c^\infty(\mathbb{R}).$$

Obviously if f is differentiable then f' is its weak derivative. Generally the taking of weak derivatives will force us to expand our notion of 'function'.

For example, for $f(t) = |t|$ we have, for any $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$,

$$\begin{aligned}\int_{\mathbb{R}} f(t)\phi'(t)\ dt &= -\int_{-\infty}^0 t\phi'(t)\ dt + \int_0^\infty t\phi'(t)\ dt \\ &= -[t\phi(t)]_{-\infty}^0 + \int_{-\infty}^0 \phi(t) + [t\phi(t)]_0^\infty - \int_0^\infty \phi(t)\ dt = -\int_{\mathbb{R}} \operatorname{sign}(t)\phi(t)\ dt\end{aligned}$$

where $\operatorname{sign}(t)$ is the function equal to -1 for negative t and equal to 1 for positive t. Notice that f is continuous but its weak derivative is not. If we want to take a

[1] As in the previous chapter, we will work with complex valued functions and complex vector bundles. Thus for instance the symbol $\mathcal{C}^\infty(M)$ will now mean $\mathcal{C}^\infty(M;\mathbb{C})$.

second weak derivative we compute

$$\int_{\mathbb{R}} \operatorname{sign}(t)\phi'(t)\, dt = -\int_0^\infty \phi'(t) + \int_0^\infty \phi'(t)\, dt = -[\phi(t)]_\infty^0 + [\phi(t)]_0^\infty = -2\phi(0).$$

This shows that the weak derivative of of $\operatorname{sign}(t)$ is a multiple of the Dirac delta 'function', δ_0, defined by

$$\int_{\mathbb{R}} \phi(t)\delta_0(t)\, dt = \phi(0), \text{ for all } \phi \in \mathcal{C}_c^\infty(\mathbb{R}).$$

Of course there is no function satisfying the definition of the delta function. Instead we should think of δ_0 as a linear functional

$$\delta_0 : \mathcal{C}_c^\infty(\mathbb{R}) \longrightarrow \mathbb{C}, \quad \delta_0(\phi) = \phi(0).$$

This is the motivating example of a distribution.

A **distribution on** $\mathbb{R}^m$ is a continuous linear functional on smooth functions of compact support,

$$T : \mathcal{C}_c^\infty(\mathbb{R}^m) \longrightarrow \mathbb{C}.$$

Continuity here means that for every compact set $K \subseteq \mathbb{R}^m$ there are constants C and ℓ satisfying

$$|T(\phi)| \leq C \sum_{|\alpha| \leq \ell} \sup_K |\partial^\alpha \phi| \text{ for all } \phi \in \mathcal{C}_c^\infty(K).$$

Similarly, let (M, g) be an oriented Riemannian[2] manifold. A **distribution on** M is a continuous linear functional on smooth functions of compact support,

$$T : \mathcal{C}_c^\infty(M) \longrightarrow \mathbb{C},$$

such that for every compact set $K \subseteq M$ there are constants C and ℓ satisfying

$$|T(\phi)| \leq C \sum_{|\alpha| \leq \ell} \sup_K |\nabla^\alpha \phi|_g \text{ for all } \phi \in \mathcal{C}_c^\infty(K).$$

We will denote[3] the set of distributions by $\mathcal{C}^{-\infty}(M)$. If the same integer ℓ can be used for all compact sets K, we say that T is of order ℓ. (A distribution of order ℓ

[2]We do not need to have a Riemannian metric or an orientation to define the space of distributions. It does make things simpler, though, since we can use the Riemannian volume form to carry out integrations (really to trivialize the 'density bundle') and the gradient to measure regularity.

[3]In the literature one often finds the set of distributions denoted $\mathcal{D}'(M)$. This is because Laurent Schwartz, who first defined distributions, denoted the smooth functions of compact support by $\mathcal{D}(M)$.

has a unique extension to a continuous linear functional on $\mathcal{C}^\ell_c(M)$, but we shall not use this.)

We can restate the continuity condition in terms of sequences: A linear functional T on $\mathcal{C}^\infty(M)$ is a distribution if and only if whenever $\phi_j \in \mathcal{C}^\infty_c(M)$ is a sequence supported in a fixed compact subset of M that is converging uniformly to zero together with all of its derivatives, we have $T(\phi_j) \longrightarrow 0$.

The Hermitian pairing that the metric induces on $\mathcal{C}^\infty(M;\mathbb{C})$ yields an inclusion

$$\begin{aligned} \mathcal{C}^\infty(M) &\longrightarrow \mathcal{C}^{-\infty}(M) \\ h &\longmapsto T_h \end{aligned}$$

$$T_h(\phi) = \langle \phi, h\rangle = \int_M \phi(p)\overline{h(p)} \,\text{dvol} \text{ for all } \phi \in \mathcal{C}^\infty_c(M).$$

indeed, any smooth function $h \in \mathcal{C}^\infty(M)$ naturally induces a distribution (of order zero) T_h

$$T_h(\phi) = \langle \phi, h\rangle = \int_M \phi(p)\overline{h(p)} \,\text{dvol} \text{ for all } \phi \in \mathcal{C}^\infty_c(M).$$

More generally, we get a distribution (of order zero) T_h, from any function h for which the pairing $\langle \phi, h\rangle$ can be guaranteed to be finite,

$$L^1_{\text{loc}}(M) = \{h \text{ measurable } : \langle \phi, h\rangle < \infty \text{ for all } \phi \in \mathcal{C}^\infty_c(M)\}.$$

Similarly, any measure on M, μ, defines a distribution T_μ of order zero

$$T_\mu(\phi) = \int_M \phi \, d\mu \text{ for all } \phi \in \mathcal{C}^\infty_c(M).$$

The converse is true as well[4], any distribution of order zero is a measure on M. An example of a distribution that is not a measure is

$$T(\phi) = \phi'(0) \text{ for all } \phi \in \mathcal{C}^\infty_c(M).$$

As a very convenient abuse of notation, the pairing between a smooth function of compact support f and a distribution T is denoted

$$\begin{gathered} \langle \cdot, \cdot \rangle : \mathcal{C}^\infty_c(M) \times \mathcal{C}^{-\infty}(M) \longrightarrow \mathbb{C} \\ \langle \phi, T\rangle = T(\phi) \end{gathered}$$

and then in another abuse we identify T_h and h.

[4] See Theorem 2.1.6 of Lars Hörmander, The analysis of linear partial differential operators. I. Distribution theory and Fourier analysis. Classics in Mathematics. Springer-Verlag, Berlin, 2003.

Now we can proceed to extend many concepts and operations of $\mathcal{C}^\infty(M)$ to $\mathcal{C}^{-\infty}(M)$ by duality using the pairing $\langle\cdot,\cdot\rangle$. For instance, we can *multiply a distribution T by a smooth function $f \in \mathcal{C}^\infty(M)$* by defining

$$\langle \phi, fT\rangle = \langle \phi\overline{f}, T\rangle, \text{ for all } \phi \in \mathcal{C}_c^\infty(M).$$

We say that a distribution T *vanishes on an open set* $\mathcal{U}$ if

$$\langle \phi, T\rangle = 0, \text{ for all } \phi \in \mathcal{C}_c^\infty(\mathcal{U}).$$

The **support of a distribution** $\operatorname{supp}(T)$ is the set of points in M having no neighborhood on which T vanishes. Equivalently, a point ζ is *not* in the support of T if there is a neighborhood of ζ on which T vanishes. The **distributions of compact support** will be denoted[5] $\mathcal{C}_c^{-\infty}(M)$. A distribution of compact support can be evaluated not just on functions of compact support but on all of $\mathcal{C}^\infty(M)$. In fact, distributions of compact support form the continuous dual of the space $\mathcal{C}^\infty(M)$.

Although it makes no sense to talk about the value of a distribution at a point, we can restrict a distribution to an open set $\mathcal{U}$ by restricting its domain to functions with compact support in $\mathcal{U}$. We say that *two distributions S and T coincide on an open set* $\mathcal{U}$ if

$$\langle \phi, S\rangle = \langle \phi, T\rangle \text{ for all } \phi \in \mathcal{C}_c^\infty(\mathcal{U}).$$

If on each set of an open cover of M we have a distribution, and these distributions coincide on overlaps, then we can use a partition of unity to construct a distribution that restricts to the given ones. It is also true that two distributions coincide if and only if they coincide on open sets covering M.

A closely related concept to the support of a distribution is that of **the singular support of a distribution** $\operatorname{sing\,supp}(T)$, defined by specifying its complement: a point $\zeta \in M$ is *not* in the singular support of T if there is a smooth function f that coincides with T in a neighborhood of ζ.

We can *apply a linear operator F to T* by using its formal adjoint F^*,

$$\langle \phi, F(T)\rangle = \langle F^*(\phi), T\rangle, \text{ for all } \phi \in \mathcal{C}_c^\infty(M).$$

In particular, we can apply a differential operator to a distribution, e.g.,

$$\langle \phi, \Delta T\rangle = \langle \Delta\phi, T\rangle, \text{ for all } \phi \in \mathcal{C}_c^\infty(M).$$

The weak derivatives we discussed above are computed in exactly this way. Thus a general function in, say, $L^1(M)$ is not differentiable as a function, but is infinitely differentiable as a distribution!

[5] Schwartz' notation was $\mathcal{E}'(M)$, and one often finds this in the literature.

If $E \longrightarrow M$ is a vector bundle over M we define **distributional sections of** E to be continuous linear functionals on $\mathcal{C}_c^\infty(M; E^*)$. As above, there is a natural way of thinking of a smooth section τ of E as a distributional section T_τ of E, namely

$$T_\tau(\omega) = \int_M \omega(\tau) \ \mathrm{dvol}_M, \text{ for all } \omega \in \mathcal{C}_c^\infty(M; E^*).$$

If we assume that E is endowed with a metric, g_E, we can identify E with E^* and identify distributional sections of E with continuous linear functionals on $\mathcal{C}_c^\infty(M; E)$. We can then extend our abuse of notation and denote the evaluation of a distribution T on a compactly supported section σ by

$$\langle \sigma, T \rangle_E = T(\sigma)$$

and also denote T_τ merely by τ.

All of the operations on distributions mentioned above extend to distributional sections. In particular we can talk about the support of a distributional section and then the distributional sections of compact support, $\mathcal{C}_c^{-\infty}(M; E)$. These distributions have a simple description

$$\mathcal{C}_c^{-\infty}(M; E) = \mathcal{C}_c^{-\infty}(M) \underset{\mathcal{C}^\infty(M)}{\otimes} \mathcal{C}^\infty(M; E).$$

Indeed, every simple tensor

$$T \otimes \sigma \in \mathcal{C}_c^{-\infty}(M) \underset{\mathcal{C}^\infty(M)}{\otimes} \mathcal{C}^\infty(M; E)$$

defines an element in the dual to $\mathcal{C}^\infty(M; E^*)$ by

$$(T \otimes \sigma)(\omega) = T(\omega(\sigma)), \text{ for all } \omega \in \mathcal{C}^\infty(M; E^*),$$

and so we have an inclusion $\mathcal{C}_c^{-\infty}(M) \otimes_{\mathcal{C}^\infty(M)} \mathcal{C}^\infty(M; E) \longrightarrow \mathcal{C}_c^{-\infty}(M; E)$. To see that this map is surjective, start with $S \in \mathcal{C}_c^{-\infty}(M; E)$ and note that S defines, given any section $\omega \in \mathcal{C}^\infty(M; E^*)$, a compactly supported distribution on M via

$$\mathcal{C}^\infty(M) \ni \phi \mapsto S(\phi\omega) \in \mathbb{C}.$$

Since the support of S is compact, we can cover it with finitely many sets $\{\mathcal{U}_i\}$ that trivialize E^*, choose $\{s_{i1}, \ldots, s_{ik}\}$ trivializing sections of E^* supported in $\mathcal{U}_i$, and then write

$$\begin{gathered} S(\omega) = S(\sum_{i,j} \omega_{ij} s_{ij}) = \sum_{i,j} S(s_{ij})(\overline{\omega(s_{ij})}) \\ = \left(\sum_{i,j} S(s_{ij}) \otimes s_{ij} \right)(\omega), \text{ for all } \omega \in \mathcal{C}^\infty(M; E^*). \end{gathered}$$

4.2 Examples

4.2.1 Fundamental solutions

Let P be a constant coefficient differential operator on $\mathbb{R}^m$, $P \in \mathrm{Diff}^k(\mathbb{R}^m)$,

$$P = \sum_{|\alpha|\leq k} a_\alpha(x) D^\alpha.$$

We say that a distribution $Q \in \mathcal{C}^{-\infty}(\mathbb{R}^m)$ is a **fundamental solution of the differential operator** P if

$$PQ = \delta_0.$$

Two fundamental solutions will differ by an element of the null space of P.

We have already seen that, in $\mathbb{R}$, $\partial_x^2 |x| = -2\delta_0$, hence

$$-\tfrac{1}{2}|x| \text{ is a fundamental solution of } \Delta = -\partial_x^2.$$

On $\mathbb{R}^m$, the fundamental solution of the Laplacian is given by

$$Q_m = \begin{cases} (2\pi)^{-1}\log|x| & \text{if } m = 2 \\ -\frac{1}{(m-2)\,\mathrm{Vol}(\mathbb{S}^{m-1})}|x|^{2-m} & \text{if } m > 2 \end{cases}$$

Indeed, note that on $|x| > 0$, $\Delta Q_m = 0$, so we can write, for any $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^m)$,

$$\begin{gathered} \langle \phi, \Delta Q_m\rangle = \langle \Delta\phi, Q_m\rangle = \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon} Q_m \Delta\phi \, dx \\ = \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon} Q_m\Delta\phi - \phi\Delta Q_m \, dx = -\lim_{\varepsilon\to 0}\int_{|x|=\varepsilon} Q_m\partial_r\phi - \phi\partial_r Q_m \, dS \\ = \begin{cases} -\lim_{\varepsilon\to 0}\int_{|x|=\varepsilon} \dfrac{\log\varepsilon\partial_r\phi - \phi\varepsilon^{-1}}{2\pi}\, dS & \text{if } m = 2 \\ -\lim_{\varepsilon\to 0}\int_{|x|=\varepsilon} \dfrac{\varepsilon^{2-m}\partial_r\phi - \phi(2-m)\varepsilon^{1-m}}{(2-m)\,\mathrm{Vol}(\mathbb{S}^{m-1})}\, dS & \text{if } m > 2 \end{cases} \\ = \phi(0) \end{gathered}$$

where in the fourth equality we have used Green's formula.

If we identify $\mathbb{R}^2$ with $\mathbb{C}$ by identifying (x, y) with $z = x + iy$ then the operator

$$\overline{\partial} = \tfrac{1}{2}(\partial_x + i\partial_y)$$

on complex valued functions has as null space the space of holomorphic functions. A similar application of Green's theorem allows us to identify $\frac{1}{\pi z}$ as a fundamental solution for $\overline{\partial}$.

4.2.2 Principal values and finite parts

Let us see a few more examples of distributions. The function $x \mapsto x^\alpha$ is locally integrable on $\mathbb{R}$ if $\alpha > -1$. For $\alpha = -1$ we define the (Cauchy) **principal value of** $\frac{1}{x}$, $\mathrm{PV}(\frac{1}{x}) \in \mathcal{C}^{-\infty}(\mathbb{R})$ by

$$\begin{aligned}\langle \phi, \mathrm{PV}(\tfrac{1}{x})\rangle &= \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon} \frac{\phi(x)}{x}\,dx = \int_0^\infty \frac{\phi(x)-\phi(-x)}{x}\,dx \\ &= \int_0^\infty \int_{-1}^1 \phi'(tx)\,dt\,dx, \qquad \text{for all } \phi \in \mathcal{C}_c^\infty(\mathbb{R}).\end{aligned}$$

This is indeed an element of $\mathcal{C}^{-\infty}(\mathbb{R})$, since whenever $\operatorname{supp}\phi \subseteq [-C, C]$ we have

$$\left|\int_0^\infty \int_{-1}^1 \phi'(tx)\,dt\,dx\right| \leq 2C \sup|\phi'|,$$

so $\mathrm{PV}(\frac{1}{x})$ is a distribution of order at most one (in fact of order one). Notice that $x\,\mathrm{PV}(\frac{1}{x}) = 1$ since, for any $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$,

$$\langle \phi, x\,\mathrm{PV}(\tfrac{1}{x})\rangle = \langle x\phi, \mathrm{PV}(\tfrac{1}{x})\rangle = \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon} \frac{\phi(x)x}{x}\,dx = \int \phi(x)\,dx = \langle \phi, 1\rangle.$$

It is then easy to see that

$$T \in \mathcal{C}^{-\infty}(\mathbb{R}), \quad xT = 1 \implies T = \mathrm{PV}(\tfrac{1}{x}) + C\delta_0, C \in \mathbb{C}.$$

so in this sense $\mathrm{PV}(\frac{1}{x})$ is a good replacement for $\frac{1}{x}$ as a distribution.

What about higher powers of x^{-1}? Simply taking the principal value will not work for x^{-2} since we have

$$\int_{|x|>\varepsilon} \frac{\phi(x)}{x^2}\,dx = \int_{x>\varepsilon} \frac{\phi(x)+\phi(-x)}{x^2}\,dx = \frac{2\phi(0)}{\varepsilon} + (\text{ bounded })$$

so instead we define the (Hadamard) **finite part of** x^{-2}, $\mathrm{FP}(\frac{1}{x^2})$ by

$$\begin{aligned}\langle \phi, \mathrm{FP}(\tfrac{1}{x^2})\rangle &= \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon} \frac{\phi(x)-\phi(0)}{x^2}\,dx = \int_0^\infty \frac{\phi(x)-2\phi(0)+\phi(-x)}{x^2}\,dx \\ &= \int_0^\infty \left(\int_0^1 s\int_{-1}^1 \phi''(stx)\,dt\,ds\right)\,dx, \text{ for all } \phi \in \mathcal{C}_c^\infty(\mathbb{R}).\end{aligned}$$

As before, this is clearly a distribution of order at most two and we have

$$T \in \mathcal{C}^{-\infty}(\mathbb{R}), \quad x^2T = 1 \implies T = \mathrm{FP}(\tfrac{1}{x^2}) + C_0\delta_0 + C_1\partial_x\delta_0, \quad C_0, C_1 \in \mathbb{C}.$$

We can continue this and define $\mathrm{FP}(\frac{1}{x^n})$ for any $n \in \mathbb{N}$, $n \geq 2$, by

$$\langle \phi, \mathrm{FP}(\tfrac{1}{x^n}) \rangle = \lim_{\varepsilon \to 0} \int_{|x|>\varepsilon} \frac{\phi - \sum_{k=0}^{n-2} \frac{1}{k!}\phi^{(k)}(0)x^k}{x^n} \, dx.$$

This defines a distribution of order at most n and moreover

$$T \in \mathcal{C}^{-\infty}(\mathbb{R}), \quad x^n T = 1 \implies T = \mathrm{FP}(\tfrac{1}{x^n}) + \sum_{k=0}^{n-1} C_i \partial_x^{(k)} \delta_0, \quad \{C_i\} \subseteq \mathbb{C}.$$

One can similarly define finite parts of non-integer powers of x and also define finite parts in higher dimensions.

4.3 Tempered distributions

On $\mathbb{R}^m$, the class of Schwartz functions $\mathcal{S}(\mathbb{R}^m)$ sits in-between the smooth functions and the smooth functions with compact support

$$\mathcal{C}_c^\infty(\mathbb{R}^m) \subseteq \mathcal{S}(\mathbb{R}^m) \subseteq \mathcal{C}^\infty(\mathbb{R}^m)$$
$$\mathcal{S}(\mathbb{R}^m) = \left\{ f \in \mathcal{C}^\infty(\mathbb{R}^m) : p_{\alpha,\beta}(f) = \sup_{\mathbb{R}^m} |x^\alpha \partial^\beta f| < \infty, \text{ for all } \alpha, \beta \right\}$$

Correspondingly the dual spaces satisfy

$$\mathcal{C}_c^{-\infty}(\mathbb{R}^m) \subseteq \mathcal{S}'(\mathbb{R}^m) \subseteq \mathcal{C}^{-\infty}(\mathbb{R}^m)$$
$$\mathcal{S}'(\mathbb{R}^m) = \left\{ T \in \mathcal{C}^{-\infty}(\mathbb{R}^m) : p_{\alpha,\beta}(f) \to 0 \text{ for all } \alpha, \beta \implies T(f_j) \to 0 \right\}.$$

The dual space of $\mathcal{S}$, $\mathcal{S}'(\mathbb{R}^m)$, is known as the space of **tempered distributions.**

Notice that these inclusions are strict. Any L^p function will be in $\mathcal{S}'(\mathbb{R}^m)$ but will only be in $\mathcal{C}_c^{-\infty}(\mathbb{R}^m)$ if it has compact support. On the other hand, the function $e^{|x|^2}$ defines a distribution in $\mathcal{C}^{-\infty}(\mathbb{R}^m)$ but not a tempered distribution.

4.3.1 Convolution

The convolution of two functions in $\mathcal{S}(\mathbb{R}^m)$ is

$$(f * h)(x) = \int_{\mathbb{R}^m} f(x-y)h(y)\, dy = \int_{\mathbb{R}^m} h(x-y)f(y)\, dy = (h * f)$$

and defines a bilinear map $\mathcal{S}(\mathbb{R}^m) \times \mathcal{S}(\mathbb{R}^m) \longrightarrow \mathcal{S}(\mathbb{R}^m)$. The commutativity is particularly clear from the expression

$$\langle \phi, f * h \rangle = \int \int f(x)h(y)\phi(x+y)\, dx\, dy,$$

which we can also write as

$$\langle \phi, f * h \rangle = \int h(y) \int f(x)\phi(x+y)\, dxdy = \int h(y) \int f(z-y)\phi(z)\, dzdy$$
$$= \langle \check{f} * \phi, h \rangle, \text{ where } \check{f}(x) = f(-x).$$

This allows us to use duality to define convolution of a tempered distribution with a Schwartz function

$$* : \mathcal{S}(\mathbb{R}^m) \times \mathcal{S}'(\mathbb{R}^m) \longrightarrow \mathcal{S}'(\mathbb{R}^m)$$
$$\langle \phi, f * S \rangle = \langle \check{f} * \phi, S \rangle \text{ for all } f, \phi \in \mathcal{S}(\mathbb{R}^m)$$

and similarly

$$* : \mathcal{C}^\infty_c(\mathbb{R}^m) \times \mathcal{C}^{-\infty}(\mathbb{R}^m) \longrightarrow \mathcal{C}^{-\infty}(\mathbb{R}^m)$$
$$\langle \phi, f * S \rangle = \langle \check{f} * \phi, S \rangle \text{ for all } f, \phi \in \mathcal{C}^\infty_c(\mathbb{R}^m).$$

For instance, notice that we have

$$\langle \phi, f * \delta_0 \rangle = \langle \check{f} * \phi, \delta_0 \rangle = \int_{\mathbb{R}^m} f(x)\phi(x)\, dx = \langle \phi, f \rangle$$

and so $f * \delta_0 = f$ for all $f \in \mathcal{S}(\mathbb{R}^m)$.

If P is any constant coefficient differential operator, and $f, h \in \mathcal{S}(\mathbb{R}^m)$ then

$$P(f * h) = P(f) * h = f * P(h)$$

and so by duality we also have

$$P(f * S) = P(f) * S = f * P(S), \text{ for all } f \in \mathcal{S}(\mathbb{R}^m), S \in \mathcal{S}'(\mathbb{R}^m).$$

Putting these last two observations together, suppose that $S \in \mathcal{S}'(\mathbb{R}^m)$ is a fundamental solution of the constant coefficient differential operator P, so that $P(S) = \delta_0$, then we have

$$P(f * S) = f * P(S) = f * \delta_0 = f, \text{ for all } f \in \mathcal{S}(\mathbb{R}^m).$$

Thus a fundamental solution for P is tantamount to a right inverse for P, even though P is generally not invertible.

In particular, with Q_m as above and $m > 2$ for simplicity,

$$u \in \mathcal{S}'(\mathbb{R}^m), f \in \mathcal{S}(\mathbb{R}^m), \text{ and } \Delta u = f \implies u = f * Q_m \in \mathcal{C}^\infty(\mathbb{R}^m)$$
$$u(x) = \int_{\mathbb{R}^m} Q_m(x-y) f(y)\, dy = \frac{1}{(2-m)\operatorname{Vol}(\mathbb{S}^{m-1})} \int_{\mathbb{R}^m} \frac{f(y)}{|x-y|^{m-2}}\, dy \tag{4.1}$$

Remark 6. We can also use convolution to prove that test functions are dense in distributions, be they $\mathcal{C}_c^{-\infty}$, $\mathcal{S}'$, or $\mathcal{C}^{-\infty}$. Indeed, first say that a sequence $\phi_n \in \mathcal{C}_c^\infty(\mathbb{R}^m)$ is an *approximate identity* if

$$\phi_n \geq 0, \quad \int_{\mathbb{R}^d} \phi_n(x)\, dx = 1, \quad \phi_n(x) = 0 \text{ if } |x| > \tfrac{1}{n},$$

and note that, as distributions, $\phi_n \to \delta_0$. Let $\chi_n \in \mathcal{C}_c^\infty(\mathbb{R}^m)$ approximate the constant function one, and notice that, for any distribution S, the sequence

$$S_n \in \mathcal{C}_c^\infty(\mathbb{R}^m), \quad S_n(x) = \chi_n(x)(\phi_n * S)(x)$$

converges as distributions to S.

4.3.2 Fourier transform

Recall that Fourier transform of a function $f \in L^1(\mathbb{R}^m)$ is defined by

$$\mathcal{F} : L^1(\mathbb{R}^m) \longrightarrow L^\infty(\mathbb{R}^m)$$
$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^m} f(x) e^{-ix\cdot\xi}\, dx$$

and satisfies for $f \in \mathcal{S}(\mathbb{R}^m)$,

$$\mathcal{F}(\partial_{x_j} f)(\xi) = i\xi_j \mathcal{F}(f)(\xi), \quad \partial_{\xi_j}\mathcal{F}(f)(\xi) = -i\mathcal{F}(x_j f)(\xi).$$

To simplify powers of i, let us define $D_s = \frac{1}{i}\partial_s$ so that these identities become

$$\mathcal{F}(D_{x_j} f)(\xi) = \xi_j \mathcal{F}(f)(\xi), \quad D_{\xi_j}\mathcal{F}(f)(\xi) = -\mathcal{F}(x_j f)(\xi),$$

and indeed for any multi-indices α, β,

$$\xi^\alpha D_\xi^\beta \mathcal{F}(f)(\xi) = (-1)^{|\beta|}\mathcal{F}(D_x^\alpha x^\beta f)(\xi). \tag{4.2}$$

In particular we have the very nice result that

$$\mathcal{F} : \mathcal{S}(\mathbb{R}_x^m) \longrightarrow \mathcal{S}(\mathbb{R}_\xi^m).$$

The formal adjoint of $\mathcal{F}$, $\mathcal{F}^*$, is defined by

$$\langle \mathcal{F}(f), h\rangle_{\mathbb{R}_\xi^m} = \langle f, \mathcal{F}^*(h)\rangle_{\mathbb{R}_x^m}, \text{ for all } f, h \in \mathcal{C}_c^\infty(\mathbb{R}^m)$$

and since we have

$$\langle \mathcal{F}(f), h\rangle_{\mathbb{R}_\xi^m} = \int_{\mathbb{R}^m} \mathcal{F}(f)(\xi)\overline{h(\xi)}\, d\xi = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m\times\mathbb{R}^m} e^{-ix\cdot\xi} f(x)\overline{h(\xi)}\, d\xi dx$$

we conclude that

$$\mathcal{F}^*(h)(x) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} h(\xi) e^{ix\cdot\xi}\, d\xi.$$

Using the fact $D_j^* = D_j$ we find

$$x^\alpha D_x^\beta \mathcal{F}^*(\phi)(x) = (-1)^{|\alpha|} \mathcal{F}^*(D_\xi^\alpha \xi^\beta \phi)(x) \tag{4.3}$$

and hence

$$\mathcal{F}^* : \mathcal{S}(\mathbb{R}^m_\xi) \longrightarrow \mathcal{S}(\mathbb{R}^m_x).$$

Proposition 11 (Fourier inversion formula)**.** *As a map on $\mathcal{S}(\mathbb{R}^m)$, the Fourier transform $\mathcal{F}$ is invertible with inverse $\mathcal{F}^*$.*

Proof. We want to show that, for any $f \in \mathcal{S}(\mathbb{R}^m)$ we have $\mathcal{F}^*(\mathcal{F}(f)) = f$, i.e.,

$$\int e^{ix\cdot\xi} \left(\int e^{-iy\cdot\xi} f(y)\, dy \right) d\xi = (2\pi)^m f(x)$$

but note that the double integral does not converge absolutely and hence we can not change the order of integration. We get around this by introducing a function $\psi \in \mathcal{S}$ with $\psi(0) = 1$, letting $\psi_\varepsilon(x) = \psi(\varepsilon x)$ and then using Lebesgue's dominated convergence theorem to see that

$$\mathcal{F}^*(h) = \lim_{\varepsilon \to 0} \mathcal{F}^*(\psi_\varepsilon h).$$

Hence the integral we need to compute is given by

$$\begin{aligned}
\lim_{\varepsilon\to 0} \int e^{ix\cdot\xi} \psi(\varepsilon\xi) \left(\int e^{-iy\cdot\xi} f(y)\, dy \right) d\xi &= \lim_{\varepsilon \to 0} \int \int e^{i(x-y)\cdot\xi} \psi(\varepsilon\xi) f(y)\, d\xi\, dy \\
= (2\pi)^{m/2} \lim_{\varepsilon\to 0} \int f(y) \widehat{\psi}\left(\tfrac{y-x}{\varepsilon}\right) \tfrac{dy}{\varepsilon^m} &= (2\pi)^{m/2} \lim_{\varepsilon \to 0} \int f(x + \varepsilon\eta) \widehat{\psi}(\eta)\, d\eta \\
&= (2\pi)^{m/2} f(x) \int \widehat{\psi}(\eta)\, d\eta.
\end{aligned}$$

If we take $\psi(x) = e^{-|x|^2/2}$, then $\widehat{\psi}(\eta) = \psi(\eta)$ and $\int \widehat{\psi}(\eta)\, d\eta = (2\pi)^{m/2}$ so the result follows. One can prove $\mathcal{F}\mathcal{F}^* = \mathrm{Id}$ similarly. □

We can extend the Fourier transform and its inverse to tempered distributions in the usual way

$$\mathcal{F}, \mathcal{F}^* : \mathcal{S}'(\mathbb{R}^m) \longrightarrow \mathcal{S}'(\mathbb{R}^m)$$
$$\langle \phi, \mathcal{F}(T) \rangle = \langle \mathcal{F}^*(\phi), T \rangle, \quad \langle \phi, \mathcal{F}^*(T) \rangle = \langle \mathcal{F}(\phi), T \rangle \text{ for all } \phi \in \mathcal{S}(\mathbb{R}^m)$$

and these are again mutually inverse and satisfy (4.2), (4.3).

In particular, we can compute that

$$\langle \phi, \mathcal{F}(\delta_0) \rangle = \langle \mathcal{F}^*(\phi), \delta_0 \rangle = \mathcal{F}^*(\phi)(0) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \phi(x)\, dx = \langle \phi, (2\pi)^{-m/2} \rangle$$

so that $\mathcal{F}(\delta_0) = (2\pi)^{-m/2}$. Then, using (4.2) and (4.3), we can deduce that

$$\mathcal{F}(D^\alpha \delta_0) = (-1)^{|\alpha|}(2\pi)^{-m/2} x^\alpha, \quad \mathcal{F}^*(1) = (2\pi)^{m/2}\delta_0, \quad \mathcal{F}^*(x^\alpha) = (-1)^{|\alpha|}(2\pi)^{m/2} D^\alpha \delta_0.$$

Another consequence of Proposition 11 is that, for any $\phi \in \mathcal{S}(\mathbb{R}^m)$,

$$\|\widehat{\phi}\|_{L^2}^2 = \langle \mathcal{F}(\phi), \mathcal{F}(\phi) \rangle = \langle \mathcal{F}^*\mathcal{F}(\phi), \phi \rangle = \langle \phi, \phi \rangle = \|\phi\|_{L^2}^2$$

thus $\mathcal{F}$ has a unique extension to an invertible map

$$\mathcal{F} : L^2(\mathbb{R}^m) \longrightarrow L^2(\mathbb{R}^m)$$

and this map is an isometry – a fact known as *Plancherel's theorem.*

The Fourier transform is a very powerful tool for understanding differential operators. Note that if $P \in \mathrm{Diff}^k(\mathbb{R}^m)$,

$$P = \sum_{|\alpha| \le k} a_\alpha(x) D^\alpha,$$

and the coefficients $a_\alpha(x)$ are bounded with bounded derivatives, then for any $\phi \in \mathcal{S}(\mathbb{R}^m)$ (or even $\mathcal{S}'(\mathbb{R}^m)$) we have

$$P\phi = \mathcal{F}^*(p(x,\xi)\mathcal{F}\phi)$$
$$\text{with } p(x,\xi) = \sum_{|\alpha| \le k} a_\alpha(x)\xi^\alpha.$$

The polynomial $p(x,\xi)$ is called **the total symbol** of P and will be denoted $\sigma_{\mathrm{tot}}(P)$. One consequence which can seem odd at first is that a differential operator acts by integration! Indeed, we have

$$(P\phi)(x) = \mathcal{F}^*(\sigma_{\mathrm{tot}}(P)\mathcal{F}\phi)(x) = \frac{1}{(2\pi)^m} \int \int e^{i(x-y)\cdot\xi} p(x,\xi)\phi(y)\, dy\, d\xi \tag{4.4}$$

although note that this integral is not absolutely convergent.

Notice the commonality between (4.4) and (4.1). In each case we have a map of the form

$$A(f)(x) = \int_{\mathbb{R}^m} \mathcal{K}_A(x,y) f(y)\, dy \tag{4.5}$$

where f and $\mathcal{K}_A$, **the integral kernel of** A, may have low regularity. In fact, this describes a very general class of maps.

Theorem 12 (Schwartz kernel theorem). *Let M and M' be manifolds, $E \longrightarrow M$ and $E' \longrightarrow M'$ vector bundles, and let*

$$\mathrm{HOM}(E, E') \longrightarrow M \times M'$$

be the bundle whose fiber at $(\zeta, \zeta') \in M \times M'$ is $\mathrm{Hom}(E_\zeta, E'_{\zeta'})$. If

$$A : \mathcal{C}^\infty_c(M; E) \longrightarrow \mathcal{C}^{-\infty}(M'; E')$$

is a bounded linear mapping, there exists

$$\mathcal{K}_A \in \mathcal{C}^{-\infty}(M \times M'; \mathrm{Hom}(E, E'))$$

known as **the Schwartz kernel of** A *or the* **integral kernel of** A *such that*

$$\langle \psi, A(\phi) \rangle = \langle \psi \boxtimes \phi, \mathcal{K}_A \rangle$$

where

$$\psi \boxtimes \phi \in \mathcal{C}^\infty_c(M \times M'), \quad (\psi \boxtimes \phi)(\zeta, \zeta') = \psi(\zeta)\phi(\zeta').$$

We will not prove this theorem[6] and we will not actually use it directly. Rather we view as an inspiration to study operators by describing their Schwartz kernels. In fact, *it is common to purposely confuse an operator and its Schwartz kernel.* Note that if the integral kernel is given by a function, say $\mathcal{K}_A \in L^1_{\mathrm{loc}}(M \times M')$, then A is given by (4.5).

A simple example is the kernel of the identity map

$$\mathcal{K}_{\mathrm{Id}} \in \mathcal{C}^{-\infty}(M \times M), \quad \mathcal{K}_{\mathrm{Id}} = \delta_{\mathrm{diag}_M}$$

and we abuse notation by writing

$$f(\zeta) = \int_M \delta_{\mathrm{diag}_M} f(\zeta') \ \mathrm{dvol}(\zeta').$$

Note that on $\mathbb{R}^m$ we can also write its integral kernel in terms of the Fourier transform

$$\mathrm{Id} = \mathcal{F}^*\mathcal{F} \implies \mathcal{K}_{\mathrm{Id}} = \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} e^{i(x-y)\cdot\xi} \, d\xi. \tag{4.6}$$

Similarly we can write the kernel of a differential operator $P \in \mathrm{Diff}^*(M)$ as

$$\mathcal{K}_P \in \mathcal{C}^{-\infty}(M \times M), \quad \mathcal{K}_P = P^* \delta_{\mathrm{diag}_M}$$

or, on $\mathbb{R}^m$,

$$\mathcal{K}_P = \frac{1}{(2\pi)^m} \int e^{i(x-y)\cdot\xi} p(x, \xi) \, d\xi. \tag{4.7}$$

[6] For a proof, see ...

4.4 Pseudodifferential operators on $\mathbb{R}^m$

4.4.1 Symbols and oscillatory integrals

In both (4.6) and (4.7) we have integral kernels of the form

$$\frac{1}{(2\pi)^m}\int e^{i(x-y)\cdot\xi}p(x,\xi)\,d\xi$$

where $p(x,\xi)$ is a polynomial in ξ of degree k and the integral makes sense as a distribution.

The way we define pseudodifferential operators is to take a similar integral, but replace $p(x,\xi)$ with a more general function. Let $\mathcal{U}\subseteq\mathbb{R}^m$ be a (regular[7]) open set and $r\in\mathbb{R}$ a real number. A function $a\in\mathcal{C}^\infty(\mathcal{U}\times\mathbb{R}^m)$ is a **symbol of order** r **on** $\mathcal{U}$,

$$a\in S^r(\mathcal{U})$$

if for all multi-indices α, β there is a constant $C_{\alpha,\beta}$ such that

$$\sup_{x\in\mathcal{U}}|D_x^\alpha D_\xi^\beta a(x,\xi)|\leq C_{\alpha,\beta}(\sqrt{1+|\xi|^2})^{r-|\beta|}.$$

Notice that the symbol condition is really only about the growth of a and its derivatives as $|\xi|\to\infty$. So we always have

$$S^r(\mathcal{U})\subseteq S^{r'}(\mathcal{U};\mathbb{R}^m)\text{ if }r<r'$$

and, if a is smooth in (x,ξ) and compactly supported (or Schwartz) in ξ, then

$$a\in\bigcap_{r\in\mathbb{R}}S^r(\mathcal{U})=S^{-\infty}(\mathcal{U};\mathbb{R}^m).$$

For example, if $r\in\mathbb{N}$ then a polynomial of degree r is an example of a symbol of order r. Also, for arbitrary $r\in\mathbb{R}$, we can take a function $a(x,\theta)$ defined for $|\theta|=1$, extend it to (x,ξ) with $|\xi|>1$ by

$$a(r,\xi)=|\xi|^r a(x,\tfrac{\xi}{|\xi|})\text{ for }|\xi|>1$$

and choose any smooth extension of a for $|\xi|<1$ and we obtain a symbol of order r.

It is easy to see that pointwise multiplication defines a map

$$S^{r_1}(\mathcal{U};\mathbb{R}^m)\times S^{r_2}(\mathcal{U};\mathbb{R}^m)\longrightarrow S^{r_1+r_2}(\mathcal{U};\mathbb{R}^m)$$

[7]Recall that an open set is regular if it is equal to the interior of its closure.

and that differentiation defines a map

$$D_x^\alpha D_\xi^\beta : S^r(\mathcal{U}) \longrightarrow S^{r-|\beta|}(\mathcal{U}; \mathbb{R}^m).$$

Given a symbol $a \in S^r(\mathcal{U})$, we will make sense of the *oscillatory integral*

$$\int e^{i(x-y)\cdot\xi} a(x,\xi)\, d\xi \tag{4.8}$$

as a distribution. If $r < -m$ then the integral converges to a continuous function (equal to $\mathcal{F}_\xi^*(a)(x, x-y)$). Similarly if $a \in S^{-\infty}(\mathcal{U})$ then the distribution is actually a smooth function. In this case, the action of this distribution on $\phi \in \mathcal{C}_c^\infty(\mathcal{U})$ is

$$\phi \mapsto \frac{1}{(2\pi)^m} \int e^{i(x-y)\cdot\xi} a(x,\xi)\phi(x)\, d\xi\, dx.$$

Notice that

$$(1 + \xi \cdot D_x) e^{i(x-y)\cdot\xi} = (1 + |\xi|^2) e^{i(x-y)\cdot\xi}$$

so for $a \in S^r$ with $r < -m$ and for any $\ell \in \mathbb{N}$ we have

$$\begin{aligned} \int e^{i(x-y)\cdot\xi} a(x,\xi)\phi(x)\, d\xi\, dx &= \int (1+|\xi|^2)^{-\ell} (1+\xi\cdot D_x)^\ell e^{i(x-y)\cdot\xi} a(x,\xi)\phi(x)\, d\xi\, dx \\ &= \int e^{i(x-y)\cdot\xi} (1-\xi\cdot D_x)^\ell ((1+|\xi|^2)^{-\ell} a(x,\xi)\phi(x))\, d\xi\, dx \end{aligned}$$

Moreover we have

$$(1-\xi\cdot D_x)^\ell ((1+|\xi|^2)^{-\ell} a(x,\xi)\phi(x)) = \sum_{|\alpha|\le \ell} a_\alpha(x,\xi) D_x^\alpha \phi(x)$$

with each $a_\alpha \in S^{r-\ell}(\mathcal{U})$. So for arbitrary $r \in \mathbb{R}$ and $a \in S^r(\mathcal{U})$, we define the distribution (4.8) on $\mathcal{C}_c^\infty(\mathcal{U})$ by

$$\phi \mapsto \frac{1}{(2\pi)^m} \int e^{i(x-y)\cdot\xi} (1-\xi\cdot D_x)^\ell ((1+|\xi|^2)^{-\ell} a(x,\xi)\phi(x))\, d\xi\, dx$$

where ℓ is any non-negative integer larger than $r+m$. We will continue to denote this distribution by (4.8).

Alternately, we can interpret (4.8) as

$$\phi \mapsto \lim_{\varepsilon\to 0} \frac{1}{(2\pi)^m} \int e^{i(x-y)\cdot\xi} \psi(\varepsilon\xi) a(x,\xi)\phi(x)\, d\xi\, dx$$

where $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^m)$ is equal to one in a neighborhood of the origin. Then we can carry out the integration by parts from the previous paragraph before taking the limit.

For any open set $\mathcal{U} \subseteq \mathbb{R}^m$, and $s \in \mathbb{R}$ we define **pseudodifferential operators on $\mathcal{U}$ of order** r in terms of their integral kernels

$$\Psi^r(\mathcal{U}) = \left\{ \mathcal{K} \in \mathcal{C}^{-\infty}(\mathcal{U} \times \mathcal{U}) : \mathcal{K} = \frac{1}{(2\pi)^m} \int e^{i(x-y)\cdot\xi} a(x,\xi)\, d\xi, \text{ for some } a \in S^r(\mathcal{U}) \right\}.$$

The symbol a is known as **the total symbol of the pseudodifferential operator**. Given $a \in S^r(\mathcal{U})$, there are several notations for the corresponding pseudodifferential operator, e.g.,

$$\mathrm{Op}(a) \text{ and } a(x,D).$$

Let us determine the singular support of $\mathcal{K} \in \Psi^r(\mathcal{U})$. The useful observation that

$$(x-y)^\alpha e^{i(x-y)\cdot\xi} = D_\xi^\alpha e^{i(x-y)\cdot\xi} \tag{4.9}$$

shows that

$$(x-y)^\alpha \mathcal{K} = \frac{1}{(2\pi)^m} \int (x-y)^\alpha e^{i(x-y)\cdot\xi} a(x,\xi)\, d\xi$$
$$= \frac{1}{(2\pi)^m} \int (D_\xi^\alpha e^{i(x-y)\cdot\xi}) a(x,\xi)\, d\xi = \frac{(-1)^{|\alpha|}}{(2\pi)^m} \int e^{i(x-y)\cdot\xi} (D_\xi^\alpha a(x,\xi))\, d\xi$$

is an absolutely convergent integral whenever $a \in S^r(\mathcal{U})$ for $r < |\alpha| - m$ and moreover taking further derivatives shows that

$$(x-y)^\alpha \mathcal{K} \in \mathcal{C}^j(\mathcal{U} \times \mathcal{U}) \text{ if } r - |\alpha| < -m - j$$

so we can conclude that $\operatorname{sing\,supp} \mathcal{K} \subseteq \mathrm{diag}_{\mathcal{U}\times\mathcal{U}}$.

As with symbols, pseudodifferential operators are filtered by degree

$$\Psi^r(\mathbb{R}^m) \subseteq \Psi^{r'}(\mathbb{R}^m) \text{ if } r < r'$$

and a special rôle is played by the operators of order $-\infty$,

$$\Psi^{-\infty}(\mathbb{R}^m) = \bigcap_{r\in\mathbb{R}} \Psi^r(\mathbb{R}^m).$$

Each $a(x,D) \in \Psi^s(\mathcal{U})$ defines a continuous linear operator

$$a(x,D) : \mathcal{C}_c^\infty(\mathcal{U}) \longrightarrow \mathcal{C}^{-\infty}(\mathcal{U})$$
$$(a(x,D)\phi)(x) = \frac{1}{(2\pi)^m} \int\int e^{i(x-y)\cdot\xi} a(x,\xi)\phi(y)\, dy\, d\xi = \mathcal{F}^*(a(x,\xi)\mathcal{F}(\phi)).$$

Proposition 13. *For $a \in S^r(\mathcal{U})$, the operator $a(x,D)$ is a continuous map*

$$a(x,D) : \mathcal{C}_c^\infty(\mathcal{U}) \longrightarrow \mathcal{C}^\infty(\mathcal{U}). \tag{4.10}$$

For $a \in S^r(\mathbb{R}^m)$, the operator $a(x,D)$ also extends to a linear operator

$$a(x,D) : \mathcal{S}(\mathbb{R}^m) \longrightarrow \mathcal{S}(\mathbb{R}^m)$$

Proof. If $a \in S^r(\mathcal{U})$ and $\phi \in \mathcal{C}_c^\infty(\mathcal{U})$ then, for every x, $a(x,\xi)\widehat{\phi}(\xi)$ is Schwartz function of ξ, hence the integral

$$a(x,D)(\phi)(x) = \int e^{ix\cdot\xi} a(x,\xi)\widehat{\phi}(\xi)\, d\xi$$

is absolutely convergent, and one can differentiate under the integral sign and the derivatives will also be given by absolutely convergent integrals. Similarly, when $a \in S^r(\mathbb{R}^m)$,

$$x^\alpha a(x,D)(\phi) = (-1)^\alpha \int (D_\xi^\alpha e^{ix\cdot\xi}) a(x,\xi)\widehat{\phi}(\xi)\, dx = (-1)^\alpha \int e^{ix\cdot\xi}(D_\xi^\alpha a(x,\xi)\widehat{\phi}(\xi))\, dx$$

is absolutely convergent for every α. □

The total symbol of a differential operator is a polynomial so it has an expansion

$$p(x,\xi) = \sum_{j=0}^k \sum_{|\beta|=j} p_\beta(x)\xi^\beta = \sum_{j=0}^k p_j(x,\xi)$$

where each $p_j(x,\xi)$ is homogeneous

$$p_j(x,\lambda\xi) = \lambda^j p_j(x,\xi)$$

and a symbol of order j.

It turns out we can 'asymptotically sum' any sequence of symbols of decreasing order.

Proposition 14 (Asymptotic completeness)**.** *Let $\mathcal{U}$ be an open subset of $\mathbb{R}^m$, $\{r_j\}$ an unbounded decreasing sequence of real numbers and $a_j \in S^{r_j}(\mathcal{U})$. There exists $a \in S^{r_0}(\mathcal{U})$ such that, for all $N \in \mathbb{N}$,*

$$a - \sum_{j=0}^{N-1} a_j \in S^{r_N}(\mathcal{U}) \tag{4.11}$$

which we denote

$$a \sim \sum a_j.$$

This is analogous to Borel's lemma to the effect that any sequence of real numbers can occur as the coefficients of the Taylor expansion of a smooth function at the origin.

Proof. Let us write $\mathcal{U}$ as the union of an increasing sequence of compact sets $K_j \subseteq \mathcal{U}$,

$$\bigcup_{j\in\mathbb{N}} K_i = \mathcal{U}, \text{ and } K_j \subseteq K_{j+1} \text{ for all } i \in \mathbb{N}.$$

Pick $\chi \in \mathcal{C}^\infty(\mathbb{R}^m, [0,1])$ vanishing on $B_1(0)$ and equal to one outside $B_2(0)$, pick a sequence converging to zero, $(\varepsilon_j) \subseteq \mathbb{R}^+$, and set

$$a(x,\xi) = \sum_{j=0}^{\infty} \chi(\varepsilon_j\xi)a_j(x,\xi).$$

Note that on any ball $\{|\xi| < R\}$ only finitely many terms of the sequence are non-zero, so the sum is a well-defined smooth function. We will see that by choosing ε_j decaying sufficiently quickly, we can arrange (5.2).

To see that $a(x,\xi) \in S^{r_0}(\mathcal{U})$ we need to show that, for each α and β there is a constant $C_{\alpha,\beta}$ such that

$$|D_x^\alpha D_\xi^\beta \chi(\varepsilon_j\xi)a(x,\xi)| \leq C_{\alpha,\beta}(\sqrt{1+|\xi|^2})^{r_0-|\beta|}.$$

Let us estimate the individual terms in the sum

$$|D_x^\alpha D_\xi^\beta \chi(\varepsilon_j\xi)a_j(x,\xi)| \leq \sum_{\gamma+\mu=\beta} |D_x^\alpha D_\xi^\gamma a_j(x,\xi)|\varepsilon_j^{|\beta|-|\mu|}|(D^{\beta-\mu}\chi)(\varepsilon_j\xi)|.$$

The term with $\gamma = \beta$ is supported in $|\xi| \leq \varepsilon_j^{-1}$ and the other terms are supported in $\varepsilon_j^{-1} \leq |\xi| \leq 2\varepsilon_j^{-1}$. We can bound the former term by

$$\begin{aligned} C_{\alpha,\beta}(a_j)(\sqrt{1+|\xi|^2})^{r_j-|\beta|} &\leq C_{\alpha,\beta}(a_j)(\sqrt{1+|\xi|^2})^{r_0-|\beta|}(\sqrt{1+\varepsilon_j^{-2}})^{r_j-r_0} \\ &\leq C_{\alpha,\beta}(a_j)\varepsilon_j^{r_0-r_j}(\sqrt{1+|\xi|^2})^{r_0-|\beta|} \end{aligned}$$

and by similar reasoning we can bound the latter terms by

$$C_{\alpha,\gamma}(a_j)\|D^{\beta-\mu}\chi\|_{L^\infty}\varepsilon_j^{r_0-r_j}(\sqrt{1+|\xi|^2})^{r_0-|\beta|}.$$

So altogether we have

$$|D_x^\alpha D_\xi^\beta \chi(\varepsilon_j\xi)a_j(x,\xi)| \leq C_{\alpha,\beta}(\chi a_j)\varepsilon_j^{r_0-r_j}(\sqrt{1+|\xi|^2})^{r_0-|\beta|}.$$

for some constant $C_{\alpha,\beta}(\chi a_j)$ independent of ε_j. Given $L \in \mathbb{N}$ we can choose a sequence $\varepsilon_j(L)$ so that

$$|\alpha| + |\beta| \leq L \implies C_{\alpha,\beta}(\chi a_j)\varepsilon_j^{r_0-r_j}(L) \leq j^{-2}$$

and hence

$$|\alpha| + |\beta| \leq L \implies |D_x^\alpha D_\xi^\beta \chi(\varepsilon_j(L)\xi)a_j(x,\xi)| \leq j^{-2}(\sqrt{1+|\xi|^2})^{r_0-|\beta|}$$

and $|D_x^\alpha D_\xi^\beta a(x,\xi)|$ satisfies the bounds we need for $|\alpha|+|\beta| \leq L$. By diagonalization (taking $\varepsilon_j = \varepsilon_j(j)$) we can arrange the convergence of

$$(\sqrt{1+|\xi|^2})^{-(r_0-|\beta|)}|D_x^\alpha D_\xi^\beta a(x,\xi)|$$

for all α, β.

The same argument applies to show that

$$\sum_{j\geq N} \chi(\varepsilon_j\xi)a_j(x,\xi) \in S^{r_N}(\mathcal{U})$$

and hence that $a \sim \sum a_j$. □

It is easy to see that if

$$a \sim \sum a_j$$

then $\{a_j\}$ determine a up to an element of $S^{-\infty}(\mathcal{U})$. Indeed, if $b \sim \sum a_j$ then

$$a - b = (a - \sum_{j=0}^{N-1} a_j) - (b - \sum_{j=0}^{N-1} a_j) \in S^{r_N}(\mathcal{U})$$

for all N, hence $a - b \in S^{-\infty}(\mathcal{U})$.

There is a further subset of symbols that imitates polynomials a little closer. We say that a symbol $a \in S^r(\mathcal{U})$ is a **classical symbol,** $a \in S^r_{cl}(\mathcal{U})$, if for each $j \in \mathbb{N}$ there is a symbol $a_{r-j} \in S^{r-j}(\mathcal{U})$ homogeneous in ξ for $|\xi| \geq 1$, i.e.,

$$a_{r-j}(x,\lambda\xi) = \lambda^{r-j}a_{r-j}(x,\xi) \text{ whenever } |\xi|, |\lambda\xi| \geq 1,$$

and we have

$$a \sim \sum_{j=1}^{\infty} a_{r-j}.$$

We refer to a_r as the **principal symbol of** $a(x, D)$.

4.4.2 Amplitudes and mapping properties

In order to prove that pseudodifferential operators are closed under composition and adjoints, it is convenient to allow the symbol to depend on both x and y. Remarkably, this does not lead to a larger class of operators.

Let $\mathcal{U} \subseteq \mathbb{R}^m$ be an open set, we say that a function $b \in \mathcal{C}^\infty(\mathcal{U} \times \mathcal{U} \times \mathbb{R}^m)$ is an **amplitude of order** $r \in \mathbb{R}$**,**

$$b \in S^r(\mathcal{U} \times \mathcal{U}; \mathbb{R}^m)$$

if for any multi-indices α, β, γ, there is a constant $C_{\alpha,\beta,\gamma}$ such that

$$\sup_{x,y\in\mathcal{U}} |D_x^\alpha D_y^\beta D_\xi^\gamma b(x,y,\xi)| \leq C_{\alpha,\beta,\gamma}(1+|\xi|)^{r-|\gamma|}. \tag{4.12}$$

Thus an amplitude is a symbol where there are twice as many 'space' variables (x, y) as there are 'cotangent' variables ξ.

An amplitude defines a distribution

$$\phi \mapsto \frac{1}{(2\pi)^m} \int e^{i(x-y)\cdot\xi} b(x,y,\xi)\phi(y)\; d\xi\; dy.$$

For fixed x this is a well-defined oscillatory integral in (y, ξ) by the discussion above. Since the estimates are uniform in x, this defines a continuous function in x. We will denote the set of distributions with amplitudes of order r by

$$\widetilde{\Psi}^r(\mathcal{U}).$$

We clearly have $\Psi^r(\mathcal{U}) \subseteq \widetilde{\Psi}^r(\mathcal{U})$, and we will soon show that $\widetilde{\Psi}^r(\mathcal{U}) = \Psi^r(\mathcal{U})$.

We start by showing this is true for operators in

$$\widetilde{\Psi}^{-\infty}(\mathcal{U}) = \bigcap_{r\in\mathbb{R}} \widetilde{\Psi}^r(\mathcal{U}).$$

Proposition 15. *The following are equivalent:*
i) $\mathcal{K} \in \Psi^{-\infty}(\mathcal{U})$,
ii) $\mathcal{K} = \frac{1}{(2\pi)^m} \int e^{i(x-y)\cdot\xi} a(x,\xi)\; d\xi$ *for some* $a \in S^{-\infty}(\mathcal{U})$,
iii) $\mathcal{K} \in \mathcal{C}^\infty(\mathcal{U} \times \mathcal{U})$ *and for every* $N \in \mathbb{N}$ *and multi indices* α, β,

$$\sup_{x,y\in\mathcal{U}} |(1+|x-y|)^N D_x^\alpha D_y^\beta \mathcal{K}(x,y)| < \infty. \tag{4.13}$$

iv) $\mathcal{K} \in \widetilde{\Psi}^{-\infty}(\mathcal{U})$.

Proof. Clearly, (*ii*) implies (*i*) and (*i*) implies (*iv*).

Let us check that (*iv*) implies (*iii*). By definition, $\mathcal{K} \in \widetilde{\Psi}^{-\infty}(\mathcal{U})$ means that, for each $r \in \mathbb{R}$, there is an amplitude $b_r \in S^r(\mathcal{U} \times \mathcal{U}; \mathbb{R}^m)$ such that

$$\mathcal{K}(x, y) = \frac{1}{(2\pi)^m} \int e^{i(x-y)\cdot\xi} b_r(x, y, \xi)\, d\xi.$$

Let us assume that $r \ll -m$ so that the integral converges absolutely and we can use (4.9) and integrate by parts to find

$$(x-y)^\alpha D_x^\beta D_y^\gamma \mathcal{K}(x, y) = (-1)^{|\alpha|} \frac{1}{(2\pi)^m} \int e^{i(x-y)\cdot\xi} (D_x + i\xi)^\beta (D_y - i\xi)^\gamma b_r(x, y, \xi)\, d\xi$$

which converges absolutely and uniformly in x, y provided $|\beta| + |\gamma| + r < |\alpha| - m$. Thus $(x-y)^\alpha D_x^\beta D_y^\gamma \mathcal{K}(x, y)$ is uniformly bounded.

Finally we show that (*iii*) implies (*ii*). If $\mathcal{K}$ satisfies (*iii*) then the function $\mathcal{L}(x, z) = \mathcal{K}(x, x - z)$ satisfies

$$\sup |(1 + |z|)^N D_x^\alpha D_z^\beta \mathcal{L}(x, z)| < \infty \text{ for all } N, \alpha, \beta.$$

Thus $\mathcal{L}(x, z)$ is a Schwartz function in z with bounded derivatives in x. We can write

$$\mathcal{K}(x, y) = \mathcal{L}(x, x - y) = \frac{1}{(2\pi)^{m/2}} \int e^{i(x-y)\cdot\xi} (\mathcal{F}_z \mathcal{L})(x, \xi)\, d\xi$$

and $\mathcal{F}_z \mathcal{L} \in S^{-\infty}(\mathcal{U})$ so we have (*ii*). □

At the moment, the mapping property (4.10) is not good enough to allow us to compose operators. We will need an extra condition on the support of the Schwartz kernels of pseudodifferential operators to guarantee that they preserve $\mathcal{C}_c^\infty(\mathcal{U})$.

Let us say that a subset $R \subseteq \mathbb{R}^m \times \mathbb{R}^m$ is a *proper support* if the left and right projections

$$\mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}^m,$$

restricted to R, are proper functions[8]. An example of a proper support is a compact set, a better example is a set R satisfying

$$R \subseteq \{(x, y, \xi) : d(x, y) \leq \varepsilon\}$$

for any fixed $\varepsilon > 0$. We will say that a distribution $\mathcal{K} \in \mathcal{C}^{-\infty}(\mathbb{R}^m \times \mathbb{R}^m)$ is **properly supported** if $\operatorname{supp} \mathcal{K}$ is a proper support. We will say that an amplitude $b(x, y, \xi) \in \mathcal{C}^\infty(\mathbb{R}^m \times \mathbb{R}^m; \mathbb{R}^m)$ is **properly supported** if

$$\operatorname{supp}_{x,y} b = \text{ closure of } \{(x, y) : b(x, y, \xi) \neq 0 \text{ for some } \xi\}$$

is a proper support.

[8]Recall that any continuous function maps compact sets to compact sets, while a function is called proper if the inverse image of a compact set is always a compact set.

Theorem 16. *Let $b(x,y,\xi) \in \mathcal{C}^\infty(\mathcal{U}\times\mathcal{U};\mathbb{R}^m)$ be a properly supported amplitude of order r. The operator*

$$B : \mathcal{C}_c^\infty(\mathcal{U}) \longrightarrow \mathcal{C}^{-\infty}(\mathcal{U})$$

$$B\phi(x) = \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m}\int_{\mathcal{U}} e^{i(x-y)\cdot\xi} b(x,y,\xi)\phi(y)\, dy\, d\xi \tag{4.14}$$

in fact maps

$$B : \mathcal{C}_c^\infty(\mathcal{U}) \longrightarrow \mathcal{C}_c^\infty(\mathcal{U})$$

and induces maps

$$B : \mathcal{C}^\infty(\mathcal{U}) \longrightarrow \mathcal{C}^\infty(\mathcal{U}), \quad B : \mathcal{C}^{-\infty}(\mathcal{U}) \longrightarrow \mathcal{C}^{-\infty}(\mathcal{U}),$$
$$\text{and } B : \mathcal{C}_c^{-\infty}(\mathcal{U}) \longrightarrow \mathcal{C}_c^{-\infty}(\mathcal{U}).$$

The formal adjoint of B is of the same form

$$B^* : \mathcal{C}_c^\infty(\mathcal{U}) \longrightarrow \mathcal{C}^{-\infty}(\mathcal{U})$$
$$B^*\phi(x) = \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m}\int_{\mathcal{U}} e^{i(x-y)\cdot\xi}\overline{b(y,x,\xi)}\phi(y)\, dy\, d\xi.$$

Moreover, $B \in \Psi^r(\mathcal{U})$ and its total symbol has an asymptotic expansion

$$a(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!}(D_\xi^\alpha D_y^\alpha b(x,y,\xi))\big|_{x=y} \tag{4.15}$$

where for a multi-index $\alpha = (\alpha_1,\dots,\alpha_m)$, $\alpha!$ denotes $\alpha_1!\cdots\alpha_m!$.

Proof. First note that the description of the adjoint follows immediately from

$$\begin{aligned}\langle B\phi, f\rangle &= \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m}\int_{\mathcal{U}}\int_{\mathcal{U}} e^{i(x-y)\cdot\xi} b(x,y,\xi)\phi(y)\overline{f(x)}\, dy\, d\xi\, dx \\ &= \frac{1}{(2\pi)^m}\int_{\mathcal{U}}\overline{\int_{\mathbb{R}^m}\int_{\mathcal{U}}\overline{e^{i(x-y)\cdot\xi} b(x,y,\xi)} f(x)}\phi(y)\, dx\, d\xi\, dy\end{aligned}$$

for all $\phi, f \in \mathcal{C}_c^\infty(\mathcal{U})$.

We see that B maps $\mathcal{C}_c^\infty(\mathcal{U})$ into $\mathcal{C}^\infty(\mathcal{U})$ by the same argument as in Proposition 13. However now if $\phi \in \mathcal{C}_c^\infty(\mathcal{U})$ and $\operatorname{supp}\phi = K$ then we know that

$$\pi_y^{-1}(K) \cap \operatorname{supp}_{x,y} b = K'$$

is a compact set and it is immediate from (4.14) that $\operatorname{supp} B(\phi) \subseteq K'$. Thus $B : \mathcal{C}_c^\infty(\mathcal{U}) \longrightarrow \mathcal{C}_c^\infty(\mathcal{U})$ and by the same argument

$$B^* : \mathcal{C}_c^\infty(\mathcal{U}) \longrightarrow \mathcal{C}_c^\infty(\mathcal{U}).$$

By duality this means that B and B^* also induce maps $\mathcal{C}^{-\infty}(\mathcal{U}) \longrightarrow \mathcal{C}^{-\infty}(\mathcal{U})$.

Let $\phi \in \mathcal{C}^\infty(\mathcal{U})$, we know that $B\phi$ is defined, but *a priori* this is a distribution. Let us check that $B\phi$ is smooth, by checking that it is smooth on any compact set $K \subseteq \mathcal{U}$. We know that

$$\pi_y^{-1}(K) \cap \operatorname{supp}_{x,y} b = K'$$

is also compact, so we can choose $\chi \in \mathcal{C}_c^\infty(\mathcal{U})$ such that $\chi(p) = 1$ for all $p \in K'$. Then, on K, we have $B\phi = B\chi\phi$ so $B\phi$ is smooth in the interior of K. This shows that

$$B : \mathcal{C}^\infty(\mathcal{U}^m) \longrightarrow \mathcal{C}^\infty(\mathcal{U}^m)$$

and since B^* is of the same form, $B^* : \mathcal{C}^\infty(\mathcal{U}) \longrightarrow \mathcal{C}^\infty(\mathcal{U})$. Then by duality both B and B^* induce maps $\mathcal{C}_c^{-\infty}(\mathcal{U}) \longrightarrow \mathcal{C}_c^{-\infty}(\mathcal{U})$.

Let us define

$$a_j(x,\xi) = \sum_{|\alpha|=j} \frac{i^{|\alpha|}}{\alpha!} (D_\xi^\alpha D_y^\alpha b(x,y,\xi))\big|_{x=y}.$$

Note that $a_j \in S^{r-j}(\mathbb{R}^m)$ so it makes sense to define $a(x,\xi)$ to be a symbol obtained by asymptotically summing $\sum a_j$. We will be done once we show that $B - a(x,D) \in \Psi^{-\infty}(\mathcal{U})$.

Recall from Taylor's theorem that, for each $\ell \in \mathbb{N}$,

$$b(x,y,\xi) = \sum_{|\alpha| \le \ell-1} \frac{1}{\alpha!} \partial_y^\alpha b(x,x,\xi)(y-x)^\alpha + \sum_{|\alpha|=\ell} c_\alpha(x,y,\xi)(y-x)^\alpha$$

$$\text{with } c_\alpha(x,y,\xi) = \int_0^1 (1-t)^{\ell-1} \partial_y^\alpha b(x,(1-t)x+ty,\xi)\, dt$$

and note the important fact that $\partial_y^\alpha b(x,y,\xi)$, and $c_\alpha(x,y,\xi)$ all satisfy (4.12). Using (4.9) and integrating by parts we find that

$$\begin{aligned}
\frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} e^{i(x-y)\cdot\xi} b(x,y,\xi)\, d\xi &= \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} e^{i(x-y)\cdot\xi} \left(\sum_{|\alpha|\le\ell-1} \frac{1}{\alpha!} D_\xi^\alpha \partial_y^\alpha b(x,x,\xi) \right) d\xi \\
&\quad + \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} e^{i(x-y)\cdot\xi} \left(\sum_{|\alpha|=\ell} \frac{1}{\alpha!} D_\xi^\alpha \partial_y^\alpha c_\alpha(x,y,\xi) \right) d\xi \\
&= \sum_{j=1}^{\ell-1} \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} e^{i(x-y)\cdot\xi} a_j(x,\xi)\, d\xi + \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} e^{i(x-y)\cdot\xi} R_\ell(x,y,\xi)\, d\xi
\end{aligned}$$

where $R_\ell(x,y,\xi)$ satisfies (4.12) with r replaced by $r-\ell$.

By definition of $a(x,\xi)$, we know that there is an element $Q_\ell \in S^{r-\ell}(\mathcal{U})$ such that

$$a(x,D) = \sum_{j=1}^{\ell-1} a_j(x,D) + Q_\ell(x,D)$$

and hence we know that

$$B - a(x,D) = \mathcal{K}(x,y) = \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m} e^{i(x-y)\cdot\xi}(R_\ell(x,y,\xi) - Q_\ell(x,\xi))\,d\xi \tag{4.16}$$

and $R_\ell(x,y,\xi) - Q_\ell(x,\xi)$ satisfies (4.12) with r replaced by $r-\ell$. It is clear from the proof of Proposition 15 that the inequalities (4.13) for a given N, α, β hold for $\mathcal{K}(x,y)$ if ℓ is large enough. Since we are able to choose ℓ as large as we like, this means that the inequalities (4.13) hold for all N, α, β, but this means that $B - a(x,D) \in \Psi^{-\infty}(\mathcal{U})$ as required. □

This theorem lets us quickly deduce that we get the same class of pseudodifferential operators using amplitudes of symbols. And it lets us show that pseudodifferential operators with properly supported amplitudes are closed under adjoints and under composition.

Corollary 17. *For all $r \in \mathbb{R}$, the sets of distributions $\Psi^r(\mathcal{U})$ and $\widetilde{\Psi}^r(\mathcal{U})$ coincide.*

Proof. Let $\mathcal{K} \in \widetilde{\Psi}^r(\mathcal{U})$ have amplitude $b(x,y,\xi)$. Choose $\chi \in \mathcal{C}^\infty(\mathcal{U}\times\mathcal{U})$ such that

$$\chi(x,y) = 0 \text{ if } d(x,y) > 1$$

then we can write $\mathcal{K}$ as the sum of $\mathcal{K}_1 \in \widetilde{\Psi}^r(\mathcal{U})$ with amplitude $\chi(x,y)b(x,y,\xi)$ and $\mathcal{K}_2 \in \widetilde{\Psi}^{-\infty}(\mathcal{U})$ with amplitude $(1-\chi(x,y))b(x,y,\xi)$. The theorem guarantees that $\mathcal{K}_1 \in \Psi^r(\mathcal{U})$ because it is properly supported, and we already know that $\widetilde{\Psi}^{-\infty}(\mathcal{U}) = \Psi^{-\infty}(\mathcal{U})$. □

We will say that a pseudodifferential operator is properly supported if its Schwartz kernel is properly supported. It is easy to see that this occurs if and only if we can represent it using a properly supported amplitude.

Corollary 18. *If $A \in \Psi^r(\mathcal{U})$ is properly supported then $A^* \in \Psi^r(\mathcal{U})$ is properly supported with symbol*

$$\sigma_{\text{tot}}(A^*)(x,\xi) \sim \sum_{|\alpha|\ge 0} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha D_x^\alpha \overline{\sigma_{\text{tot}}(A)(x,\xi)}. \tag{4.17}$$

If $\sigma_{\text{tot}}(A)$ is a classical symbol then so is $\sigma_{\text{tot}}(A^*)$ and

$$\sigma_r(A^*) = \overline{\sigma_r(A)}. \tag{4.18}$$

Proof. For any properly supported function $\chi \in \mathcal{C}^\infty(\mathcal{U} \times \mathcal{U})$ identically equal to one in a neighborhood of the diagonal containing $\operatorname{supp} \mathcal{K}_A$, the amplitude

$$\chi(x,y)\sigma_{\mathrm{tot}}(A)(x,\xi)$$

is a properly supported amplitude for A. It follows from the theorem that $A^* \in \widetilde{\Psi}^r(\mathcal{U}) = \Psi^r(\mathcal{U})$, with properly supported amplitude $\overline{\chi(y,x)\sigma_{\mathrm{tot}}(A)(y,\xi)}$, and symbol

$$\sigma_{\mathrm{tot}}(A^*) \sim \sum_{|\alpha| \geq 0} \frac{i^{|\alpha|}}{\alpha!} (D_\xi^\alpha D_y^\alpha \overline{\chi(y,x)\sigma_{\mathrm{tot}}(A)(y,\xi)})\big|_{x=y}$$

which, since $\chi(x,y)$ is equal to one in a neighborhood of the diagonal, implies (4.17). This formula in turn implies (4.18). □

Corollary 19. *If $A \in \Psi^r(\mathcal{U})$, $B \in \Psi^s(\mathcal{U})$ are properly supported, the composition*

$$A \circ B : \mathcal{C}_c^\infty(\mathcal{U}) \longrightarrow \mathcal{C}_c^\infty(\mathcal{U})$$

is a properly supported pseudodifferential operator $A \circ B \in \Psi^{r+s}(\mathcal{U})$ with symbol

$$\sigma_{\mathrm{tot}}(A \circ B) \sim \sum_{|\alpha| \geq 0} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha \sigma_{\mathrm{tot}}(A)(x,\xi) D_x^\alpha \sigma_{\mathrm{tot}}(B)(x,\xi). \tag{4.19}$$

If $\sigma_{\mathrm{tot}}(A)$ and $\sigma_{\mathrm{tot}}(B)$ are classical symbols then so is $\sigma_{\mathrm{tot}}(A \circ B)$ and

$$\sigma_{r+s}(A \circ B) = \sigma_r(A)\sigma_s(B). \tag{4.20}$$

Proof. Let $a(x,\xi) = \sigma_{\mathrm{tot}}(A)(x,\xi)$ and $b(x,\xi) = \sigma_{\mathrm{tot}}(B)(x,\xi)$.

We know that

$$B^*u(x) = \frac{1}{(2\pi)^m} \int e^{ix\cdot\xi} \overline{b(x,\xi)} \widehat{u}(\xi)\, d\xi$$

integrating against a test function $\phi \in \mathcal{C}_c^\infty(\mathcal{U})$ gives

$$\langle B\phi, u \rangle = \langle \phi, Bu \rangle = \frac{1}{(2\pi)^m} \int e^{-ix\cdot\xi} \phi(x) b(x,\xi) \overline{\widehat{u}(\xi)}\, d\xi\, dx$$

and so

$$\widehat{B\phi}(\xi) = \int e^{-iy\cdot\xi} b(y,\xi)\phi(y)\, dy.$$

This gives us a representation of B with an amplitude depending only on y. We can then write

$$(A \circ B)(\phi) = A(B\phi) = \int e^{ix\cdot\xi} a(x,\xi) \widehat{B\phi}(\xi)\, d\xi = \int e^{i(x-y)\cdot\xi} a(x,\xi) b(y,\xi) \phi(y)\, dy\, d\xi.$$

Thus $A \circ B \in \widetilde{\Psi}^{r+s}(\mathcal{U})$ with amplitude $a(x,\xi)b(y,\xi)$.

Next let us check that it is properly supported. If $K \subseteq \mathbb{R}^m$ is a compact set, then

$$K' = \pi_L(\pi_R^{-1}(K) \cap \operatorname{supp} \mathcal{K}_B) \subseteq \mathbb{R}^m \text{ is compact}$$

and hence

$$\pi_R^{-1}(K) \cap \operatorname{supp} \mathcal{K}_{A\circ B} = \pi_R^{-1}(K') \cap \operatorname{supp} A \text{ is compact.}$$

Similarly,

$$\pi_L^{-1}(K) \cap \operatorname{supp} \mathcal{K}_{A\circ B} = \pi_L^{-1}(\pi_R(\pi_L^{-1}(K) \cap \operatorname{supp} \mathcal{K}_A)) \cap \operatorname{supp} B \text{ is compact,}$$

and so $A \circ B$ is properly supported.

Thus, as before, we can multiply its amplitude by a cut-off function to make it a properly supported amplitude without affecting formula (4.15), which yields (4.19). If A and B are classical, then formula (4.19) shows that $A \circ B$ is classical, and that the principal symbols satisfy (4.20). □

To define pseudodifferential operators on manifolds, we need to know that an operator that looks like a pseudodifferential operator in one set of coordinates will look like a pseudodifferential operator in any coordinates.

Theorem 20. *Let $\mathcal{U}$ and $\mathcal{W}$ be open subsets of $\mathbb{R}^m$ and let $F : \mathcal{U} \longrightarrow \mathcal{W}$ be a diffeomorphism. Let $P \in \Psi^r(\mathcal{U})$ and let*

$$F_*P : \mathcal{C}_c^\infty(\mathcal{W}) \longrightarrow \mathcal{C}^\infty(\mathcal{W})$$
$$(F_*P)(\phi) = P(\phi \circ F) \circ F^{-1}$$

then $F_*P \in \Psi^r(\mathcal{W})$. For each multi-index α there is a function $\phi_\alpha(x,\xi)$ which is a polynomial in ξ of degree at most $|\alpha|/2$, with $\phi_0 = 1$, such that the symbol of (F_*P) satisfies

$$\sigma_{\text{tot}}(F_*P)(F(x),\xi) \sim \sum_{|\alpha|\geq 0} \frac{\phi_\alpha(x,\xi)}{\alpha!}(D_\xi^\alpha \sigma_{\text{tot}}(P))(x,(DF)^t(x)\xi). \tag{4.21}$$

If $\sigma_{\text{tot}}(P)$ is classical, so is $\sigma_{\text{tot}}(F_*P)$ and its principal symbol satisfies

$$\sigma_r(F_*P)(F(x),\xi) = \sigma_r(P)(x, DF^t(x)\xi). \tag{4.22}$$

Notice that, as with a symbol of a differential operator, this shows that the principal symbol of a pseudodifferential operator on $\mathcal{U}$ is well-defined as a section of the cotangent bundle $T^*\mathcal{U}$.

Proof. Let us denote F^{-1} by H. Note that

$$H(x) - H(y) = \int_0^1 \partial_t H(tx + (1-t)y)\, dt = \Phi(x,y) \cdot (x-y)$$

where Φ is a smooth matrix-valued function. Since $\Phi(x,x) = \lim_{y \to x} \Phi(x,y) = DH(x)$ and H is a diffeomorphism, the matrix $\Phi(x,y)$ is invertible in a neighborhood $\mathcal{Z}$ of the diagonal. Let $\chi \in \mathcal{C}_c^\infty(\mathcal{Z})$ be identically equal to one in a neighborhood of the diagonal.

Let $p = \sigma_{\mathrm{tot}}(P)$. We have

$$\begin{aligned}(F_*P)(\phi)(x) &= P(u \circ F)(H(x)) \\ &= \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} \int_{\mathcal{U}} e^{i(H(x)-y)\cdot\xi} p(H(x),\xi) u(F(y))\, dy\, d\xi \\ &= \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} \int_{\mathcal{W}} e^{i(H(x)-H(z))\cdot\xi} p(H(x),\xi) u(z) \det DH(z)\, dz\, d\xi \\ &= \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} \int_{\mathcal{W}} e^{i(x-z)\cdot\Phi(x,z)^t\xi} p(H(x),\xi) u(z) \det DH(z)\, dz\, d\xi.\end{aligned}$$

Let us briefly denote the last integrand by $\gamma(x,z,\xi)$, and decompose it as $\chi\gamma + (1-\chi)\gamma$. We know that $(1-\chi)\gamma$ is a pseudodifferential operator whose amplitude vanishes in a neighborhood of the diagonal, and hence is an element of $\Psi^{-\infty}(\mathcal{W})$. On the other hand, when integrating $\chi\gamma$ we can make the change of variables $\eta = \Phi(x,z)^t\xi$ and find

$$\frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} \int_{\mathcal{W}} e^{i(x-z)\cdot\eta} \chi(x,z) p(H(x), (\Phi(x,z)^t)^{-1}\eta) u(z) \frac{\det DH(z)}{\det \Phi(x,z)}\, dz\, d\eta$$

which is an element of $\widetilde{\Psi}^r(\mathcal{W})$ with amplitude

$$\widetilde{p}(x,z,\eta) = \frac{\chi(x,z)}{\det DF(H(z)) \det \Phi(x,z)} p(H(x), (\Phi(x,z)^t)^{-1}\eta)$$

so we can apply (4.15) and establish (4.21), which in turn proves (4.22). □

Finally, let us consider pseudodifferential operators acting between sections of bundles. In the present context this is very simple since any vector bundle over $\mathbb{R}^m$ is trivial. Thus if $E \longrightarrow \mathbb{R}^m$ and $F \longrightarrow \mathbb{R}^m$ are vector bundles, a pseudodifferential operator of order r acting between sections of E and sections of F,

$$A \in \Psi^r(\mathbb{R}^m; E, F),$$

is simply a $(\operatorname{rank} F) \times (\operatorname{rank} E)$ matrix with entries in $\Psi^r(\mathbb{R}^m)$. The extension of the discussion above of composition, adjoint, and mapping properties is straightforward.

4.5 Exercises

Exercise 1.
Show that a distribution T on $\mathbb{R}^m$ supported at the origin is a differential operator applied to δ_0.
Hint: First use continuity to find a number K such that $\langle \phi, T\rangle = 0$ for all ϕ that vanish to order K at the origin. Next, use this to view T as a linear functional on the finite dimensional vector space consisting of the coefficients of the Taylor expansion of order K of smooth functions at the origin.

Exercise 2.
Show that, if P is a differential operator and T is a distribution, $\operatorname{supp}(P(T)) \subseteq \operatorname{supp}(T)$ and $\operatorname{sing\,supp}(P(T)) \subseteq \operatorname{sing\,supp} T$.

Exercise 3.
Let $f \in \mathcal{C}^\infty(\mathbb{R})$ and let δ_0 be the Dirac delta distribution at the origin. Express the distribution $f\partial_x^3\delta_0$ as a $\mathbb{C}$-linear combination of the distributions δ_0, $\partial_x\delta_0$, $\partial_x^2\delta_0$ and $\partial_x^3\delta_0$.

Exercise 4.
a) Suppose that $T \in \mathcal{C}^{-\infty}(\mathbb{R})$ satisfies $\partial_x T = 0$ prove that T is equal to a constant $C \in \mathbb{C}$.
b) Let $S \in \mathcal{C}^{-\infty}(\mathbb{R})$, show that there is a distribution $R \in \mathcal{C}^{-\infty}(\mathbb{R})$ such that $\partial_x R = S$.
Hint: Choose $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $\int \psi \; dx = 1$. Given $\phi \in \mathcal{C}_c^\infty(M)$ show that there exists $\widetilde{\phi} \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $\phi = \left(\int \phi \, dx\right)\psi + \partial_x\widetilde{\phi}$. For (a), show that $\langle \phi, T\rangle = \left(\int \phi \, dx\right)\langle \psi, T\rangle$, so $C = \langle \psi, T\rangle$. For (b), show that $\langle \phi, R\rangle = -\langle \widetilde{\phi}, S\rangle$ defines a distribution that satisfies $\partial_x R = S$.

Exercise 5.
Suppose that $T \in \mathcal{S}'(\mathbb{R}^m)$ satisfies $\Delta T = 0$ on $\mathbb{R}^m$. Show that T is a polynomial.
Hint: First show that $\mathcal{F}(T)$ is a distribution supported at the origin.

Exercise 6.
Let $\mathcal{U} \subseteq \mathbb{R}^m$ be an open set and $\mathcal{K} \in \mathcal{C}^{-\infty}(\mathcal{U}\times\mathcal{U})$. Recall that $\mathcal{K}$ defines an operator $\mathcal{C}_c^\infty(\mathcal{U}) \longrightarrow \mathcal{C}^{-\infty}(\mathcal{U})$. Suppose that $\mathcal{K}$ satisfies

$$\mathcal{K} : \mathcal{C}_c^\infty(\mathcal{U}) \longrightarrow \mathcal{C}^\infty(\mathcal{U}) \text{ and } \mathcal{K} : \mathcal{C}_c^{-\infty}(\mathcal{U}) \longrightarrow \mathcal{C}^{-\infty}(\mathcal{U})$$

and that $\mathcal{K}$ is smooth except possibly on $\operatorname{diag}_{\mathcal{U}}$, i.e., $\operatorname{sing\,supp}\mathcal{K} \subseteq \operatorname{diag}_{\mathcal{U}}$. Show that

$$\operatorname{sing\,supp}\mathcal{K}u \subseteq \operatorname{sing\,supp} u$$

for all $u \in \mathcal{C}_c^{-\infty}(\mathcal{U})$.

Exercise 7.
Let $\mathcal{K}_A \in \mathcal{C}^{-\infty}(\mathcal{U} \times \mathcal{U})$ and suppose that the associated operator $A : \mathcal{C}_c^\infty(\mathcal{U}) \longrightarrow \mathcal{C}^{-\infty}(\mathcal{U})$ maps $\mathcal{C}_c^\infty(\mathcal{U}) \longrightarrow \mathcal{C}^\infty(\mathcal{U})$. Show that:
i)If $\mathcal{K}_A \in \mathcal{C}_c^\infty(\mathcal{U} \times \mathcal{U})$, then A extends to a map $\mathcal{C}^{-\infty}(\mathcal{U}) \longrightarrow \mathcal{C}_c^\infty(\mathcal{U})$.
ii)Conversely, if $A : \mathcal{C}^{-\infty}(\mathcal{U}) \longrightarrow \mathcal{C}_c^\infty(\mathcal{U})$ and $A^* : \mathcal{C}^{-\infty}(\mathcal{U}) \longrightarrow \mathcal{C}_c^\infty(\mathcal{U})$, show that $\mathcal{K}_A \in \mathcal{C}_c^\infty(\mathcal{U} \times \mathcal{U})$.

Exercise 8.
Give an example of a pseudodifferential operator $A \in \Psi^*(\mathbb{R}^m)$ (e.g., of order $-\infty$) and a function $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^m)$ such that $\operatorname{supp} A(\phi)$ is *not* contained in $\operatorname{supp} \phi$.

Exercise 9.
For properly supported $A \in \Psi^r(\mathbb{R}^m)$, $B \in \Psi^s(\mathbb{R}^m)$ with classical symbols, show that the commutator $[A, B] \in \Psi^{r+s-1}(\mathbb{R}^m)$ and find its principal symbol.

Exercise 10.
Assume that $A \in \Psi^r(\mathbb{R}^m)$ is properly supported with classical symbol. If $\sigma_r(A)$ is real-valued (i.e., $\sigma_r(A)(x, \xi) \in \mathbb{R}$ for all $x, \xi \in \mathbb{R}^m$). Show that $[A, A^*] = AA^* - A^*A \in \Psi^{2r-2}(\mathbb{R}^m)$.

Exercise 11.
a) Given a function

$$a(x, \xi) \in \mathcal{C}^\infty(\mathbb{R}^m \times \mathbb{S}^{m-1})$$

uniformly bounded together with its derivatives, and a number $r \in \mathbb{R}$ show that there exists a pseudodifferential operator $A \in \Psi^r(\mathbb{R}^m)$ with classical symbol such that

$$\sigma_r(A)(x, \xi)\big|_{|\xi|=1} = a(x, \xi).$$

b) If $A \in \Psi^r(\mathbb{R}^m)$ has classical symbol and $\sigma_r(A) = 0$ show that $A \in \Psi^{r-1}(\mathbb{R}^m)$.

4.6 Bibliography

Of the books we have been using so far, both *Spin geometry* by Lawson and Michelson and the second volume of *Partial differential equations* by Taylor, treat pseudodifferential operators.

For distributions, one can look at Bony, *Cours d'analyse*, volume one of Hörmander, *The analysis of linear partial differential operators*, and *Introduction to the theory of distributions* by Friedlander and Joshi.

Richard Melrose has several excellent sets of lecture notes involving pseudodifferential operators at `http://math.mit.edu/~rbm/Lecture_notes.html`. Other good sources are the books *Pseudodifferential operators* by Michael Taylor and

Pseudodifferential operators and spectral theory by Mikhail Shubin, and the lecture notes *Introduction to pseudo-differential operators* by Mark Joshi (available online on the arXiv). For an in-depth treatment, one can look at the four volumes of *The analysis of linear partial differential operators* by Hörmander.

Lecture 5

Pseudodifferential operators on manifolds

5.1 Symbol calculus and parametrices

Let M be an orientable compact Riemannian manifold, let E and F be complex vector bundles over M endowed with Hermitian metrics[1] and let us denote by

$$\mathrm{HOM}(E, F) \longrightarrow M \times M$$

the vector bundle whose fiber over a point $(\zeta, \zeta') \in M \times M$ is the vector space $\mathrm{Hom}(E_\zeta, F_{\zeta'})$. A distribution

$$\mathcal{K} \in \mathcal{C}^{-\infty}(M \times M; \mathrm{HOM}(E, F))$$

is a **pseudodifferential operator on M of order r, acting between sections of E and sections of F,**

$$\mathcal{K} \in \Psi^r(M; E, F)$$

if $\mathcal{K}$ is smooth away from the diagonal, i.e.,

$$\operatorname{sing\,supp} \mathcal{K} \subseteq \mathrm{diag}_M,$$

and, for any coordinate chart $\phi : \mathcal{U} \longrightarrow \mathcal{V}$ of M and any smooth function $\chi \in \mathcal{C}_c^\infty(\mathcal{V})$, the distribution $\chi(x)\mathcal{K}(x, y)\chi(y)$ coincides (via ϕ) with a pseudodifferential operator on $\mathcal{U}$,

$$\mathcal{K}_{\phi,\chi} \in \Psi^r(\mathcal{U}; \underline{\mathbb{R}}^{\operatorname{rank} E}, \underline{\mathbb{R}}^{\operatorname{rank} F})$$

[1] We assume orientability, compactness, and a choice of metrics for simplicity, but we could more invariantly work with operators on half-densities.

acting between the trivial bundles ϕ^*E and ϕ^*F and with *classical* symbol of order r.

Although this definition requires us to check that $\mathcal{K}_{\phi,\chi}$ is a pseudodifferential operator for every coordinate chart, Theorem 20 shows that it is enough to check this for a single atlas of M. This same theorem guarantees that the principal symbols of the operators $\mathcal{K}_{\phi,\chi}$ together define

$$\sigma_r(\mathcal{K}) \in \mathcal{C}^\infty(T^*M; \mathrm{Hom}(\pi^*E, \pi^*F))$$

with $\pi : T^*M \longrightarrow M$ the canonical projection. The principal symbol is homogeneous away form the zero section of T^*M, so we do not lose any interesting information if we restrict it to the cosphere bundle

$$\mathbb{S}^*M = \{v \in T^*M : g(v,v) = 1\}.$$

From now on, we will think of the symbol as a section

$$\sigma_r(\mathcal{K}) \in \mathcal{C}^\infty(\mathbb{S}^*M; \mathrm{Hom}(\pi^*E, \pi^*F)).$$

Let us summarize some facts about pseudodifferential operators on manifolds that follow from our study of pseudodifferential operators on $\mathbb{R}^m$.

Theorem 21. *Let M be an orientable compact Riemannian manifold and let E, F, and G be complex vector bundles over M endowed with Hermitian metrics. For each $r \in \mathbb{R}$ we have a class $\Psi^r(M; E, F)$ of distributions satisfying*

$$r < r' \implies \Psi^r(M; E, F) \subseteq \Psi^{r'}(M; E, F)$$

with a principal symbol map participating in a short exact sequence

(5.1)
$$0 \longrightarrow \Psi^{r-1}(M; E, F) \longrightarrow \Psi^r(M; E, F) \xrightarrow{\sigma_r} \mathcal{C}^\infty(\mathbb{S}^*M; \mathrm{Hom}(\pi^*E, \pi^*F)) \longrightarrow 0.$$

Pseudodifferential operators are asymptotically complete:
For any unbounded decreasing sequence of real numbers $\{r_j\}$ and $A_j \in \Psi^{r_j}(M; E, F)$ there is $A \in \Psi^{r_1}(M; E, F)$ such that, for all $N \in \mathbb{N}$,

$$A - \sum_{j=1}^{N-1} A_j \in \Psi^{r_N}(M; E, F) \tag{5.2}$$

which we denote

$$A \sim \sum A_j.$$

Pseudodifferential operators act on functions and on distributions

$$A \in \Psi^r(M; E, F) \implies$$
$$A : \mathcal{C}^\infty(M; E) \longrightarrow \mathcal{C}^\infty(M; F) \text{ and } A : \mathcal{C}^{-\infty}(M; E) \longrightarrow \mathcal{C}^{-\infty}(M; F),$$

they are closed under composition which is compatible with both the degree and the symbol

$$A \in \Psi^r(M; E, F), \quad B \in \Psi^s(M; F, G) \implies$$
$$A \circ B \in \Psi^{r+s}(M; E, G) \text{ and } \sigma_{r+s}(A \circ B) = \sigma_r(A) \circ \sigma_s(B).$$

The formal adjoint of a pseudodifferential operators is again a pseudodifferential operator of the same order and their principal symbols are mutually adjoint.

Pseudodifferential operators contain differential operators

$$\mathrm{Diff}^k(M; E, F) \subseteq \Psi^k(M; E, F)$$

and the symbol of a differential operators coincides with its symbol as a pseudodifferential operator.

Pseudodifferential operators also contain smoothing operators

$$A \in \Psi^{-\infty}(M; E, F) = \bigcap_{r \in \mathbb{R}} \Psi^r(M; E, F) \iff A \in \mathcal{C}^\infty(M \times M; \mathrm{HOM}(E, F))$$
$$\iff A : \mathcal{C}^{-\infty}(M; E) \longrightarrow \mathcal{C}^\infty(M; F) \text{ and } A^* : \mathcal{C}^{-\infty}(M; F) \longrightarrow \mathcal{C}^\infty(M; E).$$

Notice that, since M is compact, we do not need to distinguish between smooth functions and those with compact support (and similarly for distributions) and that all distributions are properly supported.

We say that a pseudodifferential operator $A \in \Psi^r(M; E, F)$ is **elliptic** if its principal symbol is invertible, i.e.,

$$\sigma_r(A) \in \mathcal{C}^\infty(M; \mathrm{Iso}(\pi^* E, \pi^* F)).$$

Thus an elliptic differential operator, such as a Laplace-type or Dirac-type operator, is also an elliptic pseudodifferential operator. Although invertibility of the symbol can not guarantee invertibility of the operator, e.g., the Laplacian on functions has all constant functions in its null space, it turns out that invertibility of the symbol is equivalent to invertibility of the operator up to smoothing operators.

Theorem 22. *A pseudodifferential operator $A \in \Psi^r(M; E, F)$ is elliptic if and only if there exists a pseudodifferential operator*

$$B \in \Psi^{-r}(M; F, E) \text{ s.t. } AB - \mathrm{Id}_F \in \Psi^{-\infty}(M; F), \quad BA - \mathrm{Id}_E \in \Psi^{-\infty}(M; E) \tag{5.3}$$

If B satisfies (5.3) we say that B is a **parametrix** for A. It is easy to see that A has a unique parametrix up to smoothing operators (see exercise 2).

Proof. Since the symbol of A is invertible and the sequence (5.1) is exact, we know that there is an operator

$$B_1 \in \Psi^{-r}(M; F, E), \text{ with } \sigma_{-r}(B) = \sigma_r(A)^{-1} \in \mathcal{C}^\infty(\mathbb{S}^*M; \mathrm{Iso}(\pi^*F, \pi^*E)).$$

The composition $A \circ B_1$ is a pseudodifferential operator of order zero and

$$\sigma_0(A \circ B_1) = \sigma_r(A) \circ \sigma_{-r}(B) = \mathrm{Id}_{\pi^*E}$$

is the same symbol as the identity operator $\mathrm{Id}_E \in \Psi^0(M; E)$. Again appealing to the exactness of (5.1) we see that

$$R_1 = \mathrm{Id}_E - AB_1 \in \Psi^{-1}(M; E).$$

Let us assume inductively that we have found $B_1, \dots, B_k$ satisfying

$$\begin{aligned} &B_i \in \Psi^{-r-(i-1)}(M; F, E) \\ \text{and } &R_k = \mathrm{Id}_E - A(B_1 + \ldots + B_k) \in \Psi^{-k}(M; E) \end{aligned} \tag{5.4}$$

and try to find $B_{k+1} \in \Psi^{-r-k}(M; F, E)$ such that

$$R_{k+1} = \mathrm{Id}_E - A(B_1 + \ldots + B_{k+1}) \in \Psi^{-k-1}(M; E).$$

Note that we can write this last equation as

$$R_{k+1} = R_k - AB_{k+1}$$

and since both R_k and AB_{k+1} are elements of $\Psi^{-k}(M; E)$ exactness of (5.1) tells us that

$$R_{k+1} \in \Psi^{-k-1}(M; E) \iff \sigma_k(R_k - AB_{k+1}) = 0.$$

So it suffices to let B_{k+1} be any element of $\Psi^{-r-k}(M; F, E)$ with principal symbol

$$\sigma_{-r-k}(B_{k+1}) = \sigma_r(A)^{-1} \circ \sigma_{-k}(R_k)$$

which we know exists by another application of exactness of (5.1). Thus inductively we can solve (5.4) for all $k \in \mathbb{N}$.

Next use asymptotic completeness to find $B \in \Psi^{-r}(M; F, E)$ such that

$$B \sim \sum B_i$$

and notice that, for any $N \in \mathbb{N}$,

$$\mathrm{Id}_E - AB = \mathrm{Id}_E - A(B_1 + \ldots + B_N) - A(B - B_1 - \ldots - B_N) \in \Psi^{-N}(M; E)$$

hence $\mathrm{Id}_E - AB \in \Psi^{-\infty}(M; E)$ as required.

The same construction can be used to construct $B' \in \Psi^{-r}(M; F, E)$ such that $\mathrm{Id}_F - B'A \in \Psi^{-\infty}(M; F)$ but then $B - B' \in \Psi^{-\infty}(M; F, E)$ by exercise 2 and so we also have $\mathrm{Id}_F - BA \in \Psi^{-\infty}(M; F)$. □

An immediate corollary is *elliptic regularity.*

Corollary 23. *Let $A \in \Psi^r(M; E, F)$ be an elliptic pseudodifferential operator. If $u \in \mathcal{C}^{-\infty}(M; E)$ then*

$$Au \in \mathcal{C}^\infty(M; F) \implies u \in \mathcal{C}^\infty(M; E).$$

More generally,

$$\operatorname{sing\,supp} Au = \operatorname{sing\,supp} u \text{ for all } u \in \mathcal{C}^{-\infty}(M; E).$$

Proof. It is true for any pseudodifferential operator A that

$$\operatorname{sing\,supp} Au \subseteq \operatorname{sing\,supp} u,$$

(see exercise 6 of the previous lecture) the surprising part is the opposite inclusion.

Let $B \in \Psi^{-r}(M; F, E)$ be a parametrix for A and $R = \mathrm{Id}_E - BA \in \Psi^{-\infty}(M; E)$ then any $u \in \mathcal{C}^{-\infty}(M; E)$ satisfies

$$u = BAu + Ru$$

and since $Ru \in \mathcal{C}^\infty(M; E)$ we have

$$\operatorname{sing\,supp} u = \operatorname{sing\,supp} BAu \subseteq \operatorname{sing\,supp} Au$$

as required. □

In the next few sections we will continue to develop consequences of the existence of a parametrix.

5.2 Hodge decomposition

An elliptic operator in $\Psi^*(M;E,F)$ induces a decomposition of the sections of E into its null space, which is finite dimensional, and its orthogonal complement and of the sections of F into its image, which has finite codimension, and its orthogonal complement. This is a version of the Hodge decomposition for an arbitrary elliptic operator. We will establish this, starting with the case of smoothing perturbations of the identity, and then show that it implies the Hodge decomposition of differential forms.

Theorem 24. *Let M be a compact Riemannian manifold, and E a Hermitian vector bundle over M. Any $R \in \Psi^{-\infty}(M;E)$ satisfies:*
i) The null space of $\operatorname{Id}_E + R$ acting on distributions is a finite dimensional space of smooth sections of E,

$$\{u \in \mathcal{C}^{-\infty}(M;E) : (\operatorname{Id}_E + R)u = 0\} \subseteq \mathcal{C}^{\infty}(M;E) \text{ is finite dimensional.}$$

ii) We have a decomposition

$$\mathcal{C}^{-\infty}(M,E) = \operatorname{Im}(\operatorname{Id}_E + R) \oplus \ker(\operatorname{Id}_E + R^*)$$

by which we mean that, whenever $f \in \mathcal{C}^{-\infty}(M;E)$,

$$f \in \operatorname{Im}(\operatorname{Id}_E + R) \iff \langle f, u\rangle_E = 0 \text{ for all } u \in \ker(\operatorname{Id}_E + R^*).$$

Proof. If $u \in \mathcal{C}^{-\infty}(M;E)$ and $(\operatorname{Id}_E + R)u = 0$ then

$$u = -Ru \in \mathcal{C}^{\infty}(M;E).$$

Since M is compact $\mathcal{C}^{\infty}(M;E) \subseteq L^2(M;E)$, so $\ker A \subseteq L^2(M;E)$. In particular, if $L \subseteq \ker A$ is a bounded subset of $L^2(M;E)$ then L is also a bounded subset of $\mathcal{C}^{\infty}(M;E)$ and so by the Arzela-Ascoli theorem every sequence has a convergent subsequence in $\mathcal{C}^{\infty}(M;E)$ and hence in $L^2(M;E)$. This means that any orthonormal set in $\ker A$ must be finite, and so $\ker A$ is a finite dimensional space.

It is easy to see that $\overline{\operatorname{Im}(\operatorname{Id}_E + R)} = (\ker(\operatorname{Id}_E + R^*))^{\perp}$ so let us show that $\operatorname{Im}(\operatorname{Id}_E + R)$ is closed. Assume that $w \in \mathcal{C}^{-\infty}(M;E)$ and there exists a sequence $(u_n) \subseteq \mathcal{C}^{-\infty}(M;E)$ such that

$$(\operatorname{Id}_E + R)u_n \to w$$

as distributions. Because smooth functions are dense in distributions (cf. Remark 6) we can assume without loss of generality that

$$(u_n) \subseteq \mathcal{C}^{\infty}(M;E) \cap (\ker(\operatorname{Id}_E + R))^{\perp}.$$

Let us first consider the case where (u_n) is a bounded subset of $L^2(M;E)$. Here, as above, we know that (Ru_n) has a convergent subsequence, but then $u_n = (\mathrm{Id}_E + R)u_n - Ru_n$ must have a convergent subsequence, say $u_n \to u$, and by continuity $w = (\mathrm{Id}_E + R)u$ as required.

On the other hand, if u_n is not bounded in L^2, then we have a subsequence with $\|u_n\|_{L^2} \to \infty$. Restricting to this subsequence and letting

$$v_n = \frac{u_n}{\|u_n\|_{L^2}}$$

we now have a sequence that is bounded in $L^2(M;E)$ and moreover $(\mathrm{Id}_E + R)v_n \to 0$. By the previous argument, v_n must have a convergent subsequence, say $v_n \to v$ and $(\mathrm{Id}_E + R)v = 0$. Since $v_n \in (\ker(\mathrm{Id}_E + R))^\perp$ this requires that $v = 0$, but then this contradicts the fact that

$$\|v\|_{L^2} = \lim_{n\to\infty} \|v_n\|_{L^2} = 1.$$

□

Corollary 25. *Let M be a compact Riemannian manifold, E and F Hermitian vector bundles over M. For any $r \in \mathbb{R}$ and any elliptic operator of order r, $A \in \Psi^r(M;E,F)$, we have:*
i) The null space of A is finite dimensional.
ii) The range of A is the orthogonal complement of the null space of A^*, i.e., if $f \in \mathcal{C}^\infty(M;F)$ then

$$f \in \mathrm{Im}(A) \iff \langle f, u\rangle_F = 0 \textit{ for all } u \in \ker(A^*)$$

An operator that satisfies parts (i) and (ii) of the corollary is called **Fredholm.**

Proof. Let $B \in \Psi^{-r}(M;F,E)$ be a parametrix for A and

$$R = BA - \mathrm{Id}_E \in \Psi^{-\infty}(M;E), \quad S = AB - \mathrm{Id}_F \in \Psi^{-\infty}(M;F).$$

The null space of A as an operator on $\mathcal{C}^{-\infty}(M;E)$ is contained in the null space of $\mathrm{Id} + R$, hence by Theorem 24 is finite dimensional and made up of smooth functions.

It is easy to see that $\mathrm{Im}\, A \subseteq (\ker A^*)^\perp$, so assume that $f \in (\ker A^*)^\perp$ and let us show that $f \in \mathrm{Im}(A)$. Using the theorem, we can write

$$(\ker A^*)^\perp = (\ker A^*)^\perp \cap \ker(\mathrm{Id}_F + S^*) \oplus (\ker A^*)^\perp \cap (\ker(\mathrm{Id}_F + S^*))^\perp \tag{5.5}$$

so we only need to show that these two summands are in $\mathrm{Im}(A)$. For the first summand, notice that $\ker(\mathrm{Id}_F + S^*) = \ker(B^*A^*)$, and we have a short exact sequence

$$0 \longrightarrow \ker A^* \longrightarrow \ker B^*A^* \xrightarrow{A^*} \mathrm{Im}(A^*) \cap \ker B^* \longrightarrow 0. \tag{5.6}$$

Since all of these spaces are finite dimensional, we can identify

$$(\ker A^*)^\perp \cap \ker(\mathrm{Id}_F + S^*) = \operatorname{Im} A\big|_{\operatorname{Im}(A^*)\cap \ker B^*} \subseteq \operatorname{Im} A.$$

For the second summand in (5.5), $\mathrm{Id}_F + S^* = B^* A^*$ implies $(\ker(\mathrm{Id}_F + S^*))^\perp \subseteq (\ker A^*)^\perp$, so we can it as

$$(\ker A^*)^\perp \cap (\ker(\mathrm{Id}_F + S^*))^\perp = (\ker(\mathrm{Id}_F + S^*))^\perp = \operatorname{Im}(\mathrm{Id}_F + S) = \operatorname{Im} AB \subseteq \operatorname{Im} A.$$

□

We can interpret this corollary, the fact that an elliptic operator $A \in \Psi^r(M; E, F)$ is Fredholm as an operator

$$A : \mathcal{C}^\infty(M; E) \longrightarrow \mathcal{C}^\infty(M; F) \text{ or } A : \mathcal{C}^{-\infty}(M; E) \longrightarrow \mathcal{C}^{-\infty}(M; F),$$

as telling us about solutions to the equation

$$Au = f.$$

Namely, we can solve this equation precisely when $f \in \mathcal{C}^{-\infty}(M; F)$ satisfies the (finitely many) conditions

$$\langle f, u_1 \rangle = 0, \langle f, u_2 \rangle = 0, \ldots, \langle f, u_k \rangle = 0 \text{ where } \{u_1, \ldots, u_k\} \text{ is a basis of } \ker A^*$$

and, in that case, there is a finite dimensional space of solutions, $u_0 + \ker A \subseteq \mathcal{C}^{-\infty}(M; E)$ with u_0 a particular solution. By elliptic regularity these solutions are smooth if and only if f is smooth.

We also know that the 'cokernel' of A as an operator on smooth sections

$$\operatorname{coker}(A) = \mathcal{C}^\infty(M; F)/A(\mathcal{C}^\infty(M; E))$$

or as an operator on distributional sections

$$\operatorname{coker}(A) = \mathcal{C}^{-\infty}(M; F)/A(\mathcal{C}^{-\infty}(M; E))$$

is naturally isomorphic to $\ker A^* \subseteq \mathcal{C}^\infty(M; F)$. The index of A as a Fredholm operator

$$\operatorname{ind}(A) = \dim \ker A - \dim \operatorname{coker} A = \dim \ker A - \dim \ker A^*$$

is finite and is the same when A acts on functions and when A acts on distributions.

Another immediate consequence of this corollary is the Hodge decomposition.

Corollary 26. *Let M be a compact Riemannian manifold, for any $k \in \mathbb{N}$ we have decompositions*

$$\mathcal{C}^\infty(M;\Lambda^k T^*M) = d\left(\mathcal{C}^\infty(M;\Lambda^{k-1}T^*M)\right) \oplus \delta\left(\mathcal{C}^\infty(M;\Lambda^{k+1}T^*M)\right) \oplus \mathcal{H}^k(M)$$
$$\mathcal{C}^{-\infty}(M;\Lambda^k T^*M) = d\left(\mathcal{C}^{-\infty}(M;\Lambda^{k-1}T^*M)\right) \oplus \delta\left(\mathcal{C}^{-\infty}(M;\Lambda^{k+1}T^*M)\right) \oplus \mathcal{H}^k(M).$$

where the latter is a topological direct sum and the former an orthogonal direct sum.

It follows that the de Rham cohomology groups coincide with the Hodge cohomology groups, and this remains true if the de Rham cohomology is computed distributionally.

Proof. Applying Corollary 25 to the Hodge Laplacian we know that

$$\begin{aligned}\mathcal{C}^\infty(M;\Lambda^k T^*M) &= \operatorname{Im}(\Delta_k) \oplus \mathcal{H}^k(M)\\ &\subseteq d\left(\mathcal{C}^\infty(M;\Lambda^{k-1}T^*M)\right) + \delta\left(\mathcal{C}^\infty(M;\Lambda^{k+1}T^*M)\right) \oplus \mathcal{H}^k(M).\end{aligned}$$

However it is easy to see that $d\left(\mathcal{C}^\infty(M;\Lambda^{k-1}T^*M)\right) \perp \delta\left(\mathcal{C}^\infty(M;\Lambda^{k+1}T^*M)\right)$ since

$$\langle d\omega, \delta\eta\rangle_{\Lambda^k T^*M} = \langle d^2\omega, \eta\rangle_{\Lambda^* T^*M} = 0,$$

which establishes the decomposition for smooth forms.

The decomposition for smooth forms shows that any distribution that vanishes on $d\left(\mathcal{C}^\infty(M;\Lambda^{k-1}T^*M)\right)$, $\delta\left(\mathcal{C}^\infty(M;\Lambda^{k+1}T^*M)\right)$, and $\mathcal{H}^k(M)$ must be identically zero. Since this is true for distributions in

$$d\left(\mathcal{C}^{-\infty}(M;\Lambda^{k-1}T^*M)\right) \cap \delta\left(\mathcal{C}^{-\infty}(M;\Lambda^{k+1}T^*M)\right),$$

we also have the decomposition for distributional forms.

The decomposition clearly shows that

$$\frac{\ker\left(d:\Omega^k(M)\longrightarrow\Omega^{k+1}(M)\right)}{\operatorname{Im}\left(d:\Omega^{k-1}(M)\longrightarrow\Omega^k(M)\right)} \cong \mathcal{H}^k(M)$$

and similarly for distributions. □

The Hodge decomposition also holds on the Dolbeault complex (3.8), and more generally for an **elliptic complex** defined as follows: Let $E_1, \ldots, E_\ell$ be Hermitian vector bundles over a compact Riemannian manifold M and suppose that we have differential operators $D_i \in \operatorname{Diff}^1(M; E_i, E_{i+1})$,

$$0 \longrightarrow \mathcal{C}^\infty(M;E_1) \xrightarrow{D_1} \mathcal{C}^\infty(M;E_2) \xrightarrow{D_2} \ldots \xrightarrow{D_{\ell-1}} \mathcal{C}^\infty(M;E_\ell) \longrightarrow 0$$

such that $D_i \circ D_{i+1} = 0$. This complex is an elliptic complex if, for all $\xi \in \mathbb{S}^*M$, the sequence of leading symbols

$$0 \longrightarrow \pi^* E_1 \xrightarrow{\sigma_1(D_1)(\xi)} \pi^* E_2 \xrightarrow{\sigma_1(D_2)(\xi)} \dots \xrightarrow{\sigma_1(D_{\ell-1})(\xi)} \pi^* E_\ell \longrightarrow 0$$

is exact. In this setting the same proof as above shows that we have a natural isomorphism of finite dimensional vector spaces

$$\frac{\ker D_i}{\operatorname{Im} D_{i-1}} \cong \ker(D_i D_{i-1}^* + D_{i+1}^* D_i),$$

i.e., the cohomology spaces of the complex are finite dimensional and coincide with the Hodge cohomology spaces.

5.3 The index of an elliptic operator

An elliptic operator acting on a closed manifold has a distinguished parametrix. Given $A \in \Psi^r(M; E, F)$ elliptic, let us write

$$\mathcal{C}^\infty(M; E) = \ker A \oplus \operatorname{Im}(A^*), \quad \mathcal{C}^\infty(M; F) = \ker A^* \oplus \operatorname{Im}(A)$$

and define $G : \mathcal{C}^\infty(M; F) \longrightarrow \mathcal{C}^\infty(M; E)$ by

$$\begin{cases} Gs = 0 \text{ if } s \in \ker A^* \\ Gs = u \text{ if } s \in \operatorname{Im}(A) \text{ and } u \text{ is the unique element of } \operatorname{Im}(A^*) \text{ s.t. } Au = s \end{cases}$$

The operator G is known as the (L^2) **generalized inverse** of A. It satisfies

$$GA = \operatorname{Id}_E - \mathcal{P}_{\ker A}, \quad AG = \operatorname{Id}_F - \mathcal{P}_{\ker A^*}$$

where $\mathcal{P}_{\ker A}$ and $\mathcal{P}_{\ker A^*}$ are the orthogonal projections onto $\ker A$ and $\ker A^*$, respectively. Notice that these projections are smoothing operators, since we can write

$$\mathcal{K}_{\mathcal{P}_{\ker A}}(x, y) = \sum_j \phi_j(x) \phi_j^*(y), \quad \text{with } \{\phi_j\} \text{ a basis of } \ker A \text{ and } \{\phi_j^*\} \text{ the dual basis} \tag{5.7}$$

and similarly for $\mathcal{P}_{\ker A^*}$. It turns out that G is itself a pseudodifferential operator, and we can express the index of A using the **trace of a smoothing operator**

$$\operatorname{Tr} : \Psi^{-\infty}(M; E) \longrightarrow \mathbb{C}$$
$$\operatorname{Tr}(P) = \int_M \operatorname{tr}_E \left(\mathcal{K}_P \big|_{\operatorname{diag}_M} \right) \operatorname{dvol}_g$$

Theorem 27. *Let $A \in \Psi^r(M; E, F)$ be an elliptic operator and $G : \mathcal{C}^\infty(M; F) \longrightarrow \mathcal{C}^\infty(M; E)$ its generalized inverse, then $G \in \Psi^{-r}(M; F, E)$ and*

$$\operatorname{ind}(A) = \operatorname{Tr}(\mathcal{P}_{\ker A}) - \operatorname{Tr}(\mathcal{P}_{\ker A^*}) =: \operatorname{Tr}([A, G]).$$

Proof. Let $B \in \Psi^{-r}(M; F, E)$ be a parametrix for A and

$$R = BA - \operatorname{Id}_E \in \Psi^{-\infty}(M; E), \quad S = AB - \operatorname{Id}_F \in \Psi^{-\infty}(M; F).$$

Making use of associativity of multiplication, we have

$$BAG = G + RG = B - B\mathcal{P}_{\ker A^*}, \quad GAB = G + GS = B - \mathcal{P}_{\ker A}$$
$$\text{hence } G = B - B\mathcal{P}_{\ker A^*} - RG \text{ and } G = B - \mathcal{P}_{\ker A}B - GS,$$

and substituting the latter into the former,

$$G = B - B\mathcal{P}_{\ker A^*} - RB + R\mathcal{P}_{\ker A}B + RGS.$$

This final term satisfies

$$RGS : \mathcal{C}^{-\infty}(M; F) \longrightarrow \mathcal{C}^\infty(M; E), \quad S^*G^*R^* : \mathcal{C}^{-\infty}(M; E) \longrightarrow \mathcal{C}^\infty(M : F)$$

which shows that $RGS \in \Psi^{-\infty}(M; F, E)$, and so $G - B \in \Psi^{-\infty}(M; F, E)$ and $G \in \Psi^{-r}(M; F, E)$.

Now note that, from (5.7),

$$\operatorname{Tr}(\mathcal{P}_{\ker A}) = \sum \|\phi_j\|^2_{L^2(M;E)} = \dim \ker A$$

and similarly for $\mathcal{P}_{\ker A^*}$ so the formula

$$\operatorname{ind}(A) = \operatorname{Tr}(\mathcal{P}_{\ker A}) - \operatorname{Tr}(\mathcal{P}_{\ker A^*})$$

is immediate. □

Let us establish a couple of properties of the trace of smoothing operators that will be useful for understanding the index.

Proposition 28. *Let M be a compact Riemannian manifold and E, F Hermitian vector bundles.*
i) The operators of finite rank

$$\{K \in \Psi^{-\infty}(M; E, F) : \dim \operatorname{Im}(K) < \infty\}$$

are dense in $\Psi^{-\infty}(M; E, F)$.
ii) If $P \in \Psi^{-\infty}(M; F, E)$ and $A \in \Psi^r(M; E, F)$ then

$$\operatorname{Tr}(AP) = \operatorname{Tr}(PA).$$

Proof. i) Given $P \in \Psi^{-\infty}(M; E, F)$ so that $\mathcal{K}_P \in \mathcal{C}^\infty(M \times M; \mathrm{HOM}(E, F))$ let us show that we can approximate $\mathcal{K}_P$ in the $\mathcal{C}^\infty$-topology by finite rank operators. Using partitions of unity we can assume that $\mathcal{K}_P$ is supported in an open set of $M \times M$ over which E and F are trivialized, so $\mathcal{K}_P$ is just a smooth matrix valued function on an open set of $\mathbb{R}^m \times \mathbb{R}^m$. Identifying this open set with an open set in $\mathbb{T}^m \times \mathbb{T}^m$ with $\mathbb{T}^m$ the m-dimensional torus, we can expand each of the entries in a Fourier series in each variable. The truncated series then are finite rank operators that converge uniformly with all of their derivatives to $\mathcal{K}_P$, and restricting their support we get finite rank operators on M that approximate P.

ii) By continuity we can assume that P has finite rank and by linearity that P has rank one, say

$$P\phi = v(\phi)s, \text{ for some } v \in \mathcal{C}^\infty(M; F^*), s \in \mathcal{C}^\infty(M; E).$$

Then we have

$$AP(\phi) = v(\phi)As, \quad PA(\phi) = v(A\phi)s$$
$$\text{so } \operatorname{Tr}(AP) = \int v(As) \, \mathrm{dvol}_g = \operatorname{Tr}(PA).$$

□

Now we can derive some of the most important properties of the index of an elliptic operator.

Theorem 29. *Let M be a compact Riemannian manifold, E, F, and G Hermitian vector bundles over M, and let $A \in \Psi^r(M; E, F)$ and $B \in \Psi^s(M; F, G)$ be elliptic.*
i) $B \circ A \in \Psi^{r+s}(M; E, G)$ is elliptic and

$$\operatorname{ind}(B \circ A) = \operatorname{ind}(B) + \operatorname{ind}(A).$$

ii) If $Q \in \Psi^{-r}(M; F, E)$ is any parametrix of A then

$$\operatorname{ind}(A) = \operatorname{Tr}(\mathrm{Id}_E - QA) - \operatorname{Tr}(\mathrm{Id}_F - AQ) =: \operatorname{Tr}([A, Q]).$$

iii) If $P \in \Psi^{-\infty}(M; E, F)$ then $\operatorname{ind}(A) = \operatorname{ind}(A + P)$.
iv) The index is a smooth homotopy invariant of elliptic operators.
v) The index of A only depends on $\sigma_r(A)$.

Proof. i) Recall that we have a short exact sequence (5.6)

$$0 \longrightarrow \ker A^* \longrightarrow \ker B^* A^* \xrightarrow{A^*} \operatorname{Im}(A^*) \cap \ker B^* \longrightarrow 0$$

as these are all finite dimensional, we can conclude that

$$\dim\ker(A^*B^*) = \dim\ker(B^*) + \dim((\ker A^*) \cap (\operatorname{Im} B^*)).$$

By the same reasoning

$$\dim\ker(BA) = \dim\ker(A) + \dim((\ker B) \cap (\operatorname{Im} A))$$

so that $\operatorname{ind}(BA)$ is

$$\begin{aligned}
&\dim\ker A + \dim((\ker B) \cap (\operatorname{Im} A)) - \dim\ker(B^*) - \dim((\ker A^*) \cap (\operatorname{Im} B^*)) \\
&= \dim\ker A + \dim((\ker B) \cap (\ker A^*)^\perp) - \dim\ker(B^*) - \dim((\ker A^*) \cap (\ker B)^\perp) \\
&= \dim\ker A + \dim(\ker B) - \dim\ker(B^*) - \dim(\ker A^*) = \operatorname{ind}(A) + \operatorname{ind}(B)
\end{aligned}$$

as required.

ii) From Theorem 27, we know that if G is the generalized inverse of A then

$$\operatorname{Tr}([A, G]) = \operatorname{ind}(A)$$

and from the proof of Theorem 27 we know that $G - Q \in \Psi^{-\infty}(M; F, E)$. It follows that

$$\begin{aligned}
&\operatorname{Tr}([A, Q]) - \operatorname{Tr}([A, G]) \\
&= \operatorname{Tr}(\operatorname{Id}_E - QA) - \operatorname{Tr}(\operatorname{Id}_F - AQ) - \operatorname{Tr}(\operatorname{Id}_E - GA) + \operatorname{Tr}(\operatorname{Id}_F - AG) \\
&= \operatorname{Tr}((G - Q)A) - \operatorname{Tr}(A(G - Q))
\end{aligned}$$

which vanishes by Proposition 28.

iii) follows immediately from (ii) since any parametrix of A is a parametrix of $A + P$.

iv) If $t \mapsto A_t \in \Psi^r(M; E, F)$ is a smooth family of elliptic pseudodifferential operators then we can construct a smooth family $t \mapsto Q_t \in \Psi^{-r}(M; F, E)$ of parametrices and hence

$$\operatorname{ind}(A_t) = \operatorname{Tr}([A_t, Q_t])$$

is a smooth function of t. Since this is valued in the integers, it must be constant and independent of t.

v) It suffices to note that if two operators A and A' have the same elliptic symbol then $tA + (1 - t)A'$ is a smooth family of elliptic operators connecting A and A', so by (iv) they have the same index.

□

5.4 Pseudodifferential operators acting on L^2

So far, we have been studying pseudodifferential operators as maps

$$\mathcal{C}^\infty(M;E) \longrightarrow \mathcal{C}^\infty(M;F) \text{ or } \mathcal{C}^{-\infty}(M;E) \longrightarrow \mathcal{C}^{-\infty}(M;F),$$

now we want to look at more refined mapping properties by looking at distributions with some intermediate regularity, e.g.,

$$\mathcal{C}^\infty(M;E) \subseteq L^2(M;E) \subseteq \mathcal{C}^{-\infty}(M;E).$$

Let us recall some definitions for linear operators acting between Hilbert spaces[2], $\mathcal{H}_1$ and $\mathcal{H}_2$. It is convenient, particularly when differential operators are involved, to allow for linear operators to be defined on a subspace of $\mathcal{H}_1$,

$$P : \mathcal{D}(P) \subseteq \mathcal{H}_1 \longrightarrow \mathcal{H}_2.$$

We think of a **linear operator between Hilbert spaces** as the pair $(P, \mathcal{D}(P))$ and only omit $\mathcal{D}(P)$ when it is clear from context. We will always assume that $\mathcal{D}(P)$ is dense in $\mathcal{H}_1$.

Thus for instance any pseudodifferential operator $A \in \Psi^r(M;E,F)$ defines a linear operator between L^2 spaces with domain $\mathcal{C}^\infty(M;E)$,

$$A : \mathcal{C}^\infty(M;E) \subseteq L^2(M;E) \longrightarrow L^2(M;F).$$

Of course, with this domain A actually maps into $\mathcal{C}^\infty(M;F)$, so we will be interested in extending A to a larger domain.

An **extension of a linear operator** $(P, \mathcal{D}(P))$ is an operator $(\widetilde{P}, \mathcal{D}(\widetilde{P}))$,

$$\widetilde{P} : \mathcal{D}(\widetilde{P}) \subseteq \mathcal{H}_1 \longrightarrow \mathcal{H}_2$$

such that $\mathcal{D}(P) \subseteq \mathcal{D}(\widetilde{P})$ and $\widetilde{P}u = Pu$ for all $u \in \mathcal{D}(P)$.

An operator $(P, \mathcal{D}(P))$ is **bounded** if it sends bounded sets to bounded sets. In particular, there is a constant C such that

$$\|Pu\|_{\mathcal{H}_2} \leq C\|u\|_{\mathcal{H}_1} \text{ for all } u \in \mathcal{D}(P)$$

and the smallest constant is called the **operator norm of** P and denoted

$$\|P\|_{\mathcal{H}_1 \longrightarrow \mathcal{H}_2}.$$

[2]For simplicity, we will focus on Hilbert spaces based on $L^2(M;E)$, but pseudodifferential operators also have good mapping properties on Banach spaces such as $L^p(M;E)$, $p \neq 2$.

Recall that an operator $(P, \mathcal{D}(P))$ is continuous at a point if and only if it is continuous at every point in $\mathcal{D}(P)$ if and only if it is bounded. A bounded operator has a unique extension to an operator defined on all of $\mathcal{H}_1$, moreover this extension is bounded with the same operator norm. If an operator is bounded, we will assume that its domain is $\mathcal{H}_1$ and generally omit it from the notation. The set of bounded operators between $\mathcal{H}_1$ and $\mathcal{H}_2$ is denoted

$$\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2).$$

A **compact operator** is an operator that sends bounded sets to pre-compact sets, and is in particular necessarily bounded. We denote the space of compact operators between $\mathcal{H}_1$ and $\mathcal{H}_2$ by

$$\mathcal{K}(\mathcal{H}_1, \mathcal{H}_2).$$

An example of a smoothing operator is a smoothing operator. Indeed, for smoothing operators between sections of E and sections of F, the image of a bounded subset of $L^2(M;E)$ is bounded in $\mathcal{C}^\infty(M;F)$ and hence by Arzela-Ascoli compact in $L^2(M;F)$.

Lemma 30. *Let M be a Riemannian manifold and E, F Hermitian vector bundles. If $R \in \Psi^{-\infty}(M;E,F)$, then $(R, \mathcal{C}^\infty(M;E))$ is a compact operator between $L^2(M;E) \longrightarrow L^2(M;F)$. Its unique extension to $L^2(M;E)$ is the restriction of R as an operator on $\mathcal{C}^{-\infty}(M;E)$.*

Since the identity map is trivially bounded, we can use this lemma to see that any operator of the form $\mathrm{Id}_E + R$ with $R \in \Psi^{-\infty}(M;E)$ defines a bounded operator on $L^2(M;E)$. There is a clever way of using this to show that any pseudodifferential operator of order zero defines a bounded operator on L^2.

Proposition 31. *If $r \leq 0$ and $P \in \Psi^r(M;E,F)$ then $(P, \mathcal{C}^\infty(M;E))$ defines a bounded linear operator between $L^2(M,E)$ and $L^2(M,F)$.*

Proof. We follow Hörmander and reduce to the smoothing case as follows. Assume we can find $Q \in \Psi^0(M,E)$, $C > 0$, and $R \in \Psi^{-\infty}(M,E)$ such that

$$P^*P + Q^*Q = C\,\mathrm{Id} + R.$$

Then we have

$$\|Pu\|^2_{L^2(M,F)} = \langle Pu, Pu\rangle_F = \langle P^*Pu, u\rangle_E$$
$$\leq \langle P^*Pu, u\rangle + \langle Q^*Qu, u\rangle = C\langle u, u\rangle + \langle Ru, u\rangle \leq (C + \|R\|_{L^2(M;E)\to L^2(M;E)})\|u\|^2_{L^2(M;E)},$$

so the proposition is proved once we construct Q, C, and R.

We proceed iteratively. Let $\sigma_0(P)$ denote the symbol of degree zero of P, e.g., $\sigma_0(P) = 0$ if $r < 0$. Choose $C > 0$ large enough so that $C - \sigma_0(P)^*\sigma_0(P)$ is, at each point of S^*M, a positive element of $\hom(E)$, then choose Q_0' so that its symbol is a positive square root of $C - \sigma_0^*(P)\sigma_0(P)$, and let $Q_0 = \frac{1}{2}(Q_0' + (Q_0')^*)$. Note that Q_0 is a formally self-adjoint elliptic operator with the same symbol as Q_0'.

Suppose we have found formally self-adjoint operators $Q_0, \dots, Q_N$ satisfying $Q_i \in \Psi^{-i}(M; E)$ and

$$P^*P + \left(\sum_{i=0}^{N} Q_i\right)^2 = C\,\mathrm{Id} - R_N, \quad R_N \in \Psi^{-1-N}(M; E).$$

Then, if $Q'_{N+1} \in \Psi^{-N-1}(M, E)$, we have

$$\begin{aligned} P^*P + \left(\sum_{i=0}^{N} Q_i + Q'_{N+1}\right)^2 - C\,\mathrm{Id} \\ = -R_N + Q'_{N+1}\left(\sum_{i=0}^{N} Q_i\right) + \left(\sum_{i=0}^{N} Q_i\right) Q'_{N+1}, \end{aligned}$$

so in order for this to be in $\Psi^{-2-N}(M; E)$, Q'_{N+1} must satisfy

$$\sigma(Q'_{N+1}Q_0 + Q_0 Q'_{N+1}) = \sigma(R_N).$$

We can always solve this equation because $\sigma(Q_0)$ is self-adjoint and positive as shown in the following lemma.

Lemma 32. *Let A be a positive self-adjoint $k \times k$ matrix, and let*

$$\Phi_A(B) = AB + BA,$$

then Φ_A is a bijection from the space of self-adjoint matrices to itself.

Proof. We can find a matrix S such that $S^* = S^{-1}$ and $S^*AS = D$ is diagonal. Define $\Psi(B) = S^*BS$ and note that $\Psi^{-1}\Phi_A\Psi = \Phi_{\Psi(A)}$ so we may replace A with $D = \mathrm{diag}(\lambda_1, \dots, \lambda_k)$. Let E_{st} denote the matrix with entries

$$(E_{st})_{ij} = \begin{cases} 1 \text{ if } s = i, t = j \\ 0 \text{ otherwise} \end{cases}$$

and note that

$$\Phi_D(E_{st} + E_{ts}) = (\lambda_s + \lambda_t)(E_{st} + E_{ts}).$$

Hence the matrices $\{E_{st} + E_{ts}\}$ with $s \leq t$ form an eigenbasis of Φ_D on the space of self-adjoint $k \times k$ matrices. Since, by assumption, each $\lambda_s > 0$ it follows that Φ_D is bijective. □

So using the lemma we can find $\sigma(Q'_{N+1})$ among self-adjoint matrices and then define $Q_{N+1} = \frac{1}{2}(Q'_{N+1} + (Q'_{N+1})^*)$, finishing the iterative step. Finally, we define Q by asymptotically summing the Q_i and the result follows. □

Corollary 33. *Let $A \in \Psi^r(M; E, F)$, $r \leq 0$, and let*

$$\|\sigma_0(A)\|_\infty = \sup_{S^*M} |\sigma_0(A)(\zeta, \xi)|,$$

then for any $\varepsilon > 0$ there is a non-negative self-adjoint operator $R_\varepsilon \in \Psi^{-\infty}(M, E)$ such that, for any $u \in L^2(M, E)$

$$\|Au\|^2_{L^2(M,F)} \leq (\|\sigma_0(A)\|^2_\infty + \varepsilon)\|u\|^2_{L^2(M,E)} + \langle R_\varepsilon u, u\rangle_E. \tag{5.8}$$

In particular, if $r < 0$ then $A \in \Psi^r(M; E, F)$ defines a compact operator $(A, \mathcal{C}^\infty(M; E))$ between $L^2(M; E)$ and $L^2(M; F)$.

Proof. The existence of R_ε satisfying the inequality (5.8) follows directly from the proof of the theorem.

If $\|\sigma_0(A)\|_\infty = 0$, we show that A is a compact operator by proving that the image of a bounded sequence is Cauchy. Let (u_j) be a bounded sequence in $L^2(M; E)$, say $\|u_j\| \leq C$ for all $j \in \mathbb{N}$. For each $n \in \mathbb{N}$ we know that $(R_{\frac{1}{n}} u_j)$ has a convergent subsequence in $L^2(M; E)$ and so by a standard diagonalization argument we can find a subsequence (u'_j) of (u_j) such that $(R_{\frac{1}{n}} u'_j)$ converges for all n.

It follows that (Au'_j) is a Cauchy sequence in $L^2(M; F)$. Indeed, given $\eta > 0$ choose $N \in \mathbb{N}$ such that $j, k > N$ implies

$$\|R_\eta(u'_j - u'_k)\|_{L^2(M;E)} \leq \eta.$$

It follows that for such j, k

$$\|A(u'_j - u'_k)\|^2_{L^2(M;F)} \leq \eta\|u'_j - u'_k\|_{L^2(M;E)} + \langle R_\eta(u'_j - u'_k), (u'_j - u'_k)\rangle_{L^2(M;E)} \leq \eta(2C) + \eta C.$$

Hence (Au'_j) is Cauchy and A is compact. □

The **graph of a linear operator** $(P, \mathcal{D}(P))$ is the subspace of $\mathcal{H}_1 \times \mathcal{H}_2$ given by

$$\{(u, Pu) \in \mathcal{H}_1 \times \mathcal{H}_2 : u \in \mathcal{D}(P)\}.$$

We say that a linear operator $(P, \mathcal{D}(P))$ is **closed** if and only if its graph is a closed subspace of $\mathcal{H}_1 \times \mathcal{H}_2$. Any bounded operator has a closed graph, and the *closed graph theorem* says that an closed operator $(P, \mathcal{D}(P))$ with $\mathcal{D}(P) = \mathcal{H}_1$ is bounded.

Let us point out that a closed operator *need not* have closed domain or closed range, however the null space of a closed operator does form a closed set.

It is very convenient to deal with closed operators, especially when studying the spectrum of an operator or dealing with adjoints. For instance, the spectrum of a linear operator that is not closed consists of the entire complex plane!

If $A \in \Psi^r(M; E, F)$ then $(A, \mathcal{C}^\infty(M; E))$ has at least two closed extensions with domains: the **maximal domain** comes from restricting the action of A as a continuous operator on distributions,

$$\mathcal{D}_{\max}(A) = \{u \in L^2(M; E) : Au \in L^2(M; F)\},$$

the **minimal domain** comes from extending the action of A as a continuous operator on smooth functions,

$$\mathcal{D}_{\min}(A) = \{u \in L^2(M; E) : \\ \text{there exists } (u_n) \subseteq \mathcal{C}^\infty(M; E) \text{ s.t. } \lim u_n = u \text{ and } (Au_n) \text{ converges}\}.$$

In the latter case, we define $Au = \lim Au_n$ and this is independent of the choice of sequence u_n as long as $\lim u_n = u$ and Au_n converges. Since the graph of $(A, \mathcal{D}_{\min}(A))$ is the closure of the graph of $(A, \mathcal{C}^\infty(M; E))$, it is clearly the minimal closed extension of A and $(A, \mathcal{D}_{\max}(A))$ is also clearly the maximal closed extension of A.

There are operators for which $\mathcal{D}_{\min}(A) \neq \mathcal{D}_{\max}(A)$ (see exercise 8). However these must have positive order, since we have shown that operators of order zero are bounded (which implies that $\mathcal{D}_{\min}(A) = \mathcal{D}_{\max}(A) = L^2(M; E)$). Next we show that elliptic operators also have a unique closed extension.

Proposition 34. *Let M be a compact Riemannian manifold and E, F Hermitian vector bundles over M, and $r > 0$. If $A \in \Psi^r(M; E, F)$ is an elliptic operator then*

$$\mathcal{D}_{\min}(A) = \mathcal{D}_{\max}(A). \tag{5.9}$$

In fact the following stronger statement is true: For any $Q \in \Psi^{-r}(M; F, E)$,

$$\operatorname{Im}(Q : L^2(M; F) \longrightarrow L^2(M; E)) \subseteq \mathcal{D}_{\min}(A). \tag{5.10}$$

Proof. Let us start by proving (5.10). Let $u \in L^2(M; F)$ and $v = Qu$, we need to show that $v \in \mathcal{D}_{\min}(A)$. Choose $(u_n) \subseteq \mathcal{C}^\infty(M; F)$, a sequence converging to u, and note that, since Q and AQ are bounded operators on $L^2(M; F)$, we have

$$v_n = Qu_n \to v, \quad Av_n = AQu_n \to Av.$$

This implies $v \in \mathcal{D}_{\min}(A)$ as required.

Next, to prove (5.9), let $B \in \Psi^{-r}(M; F, E)$ be a parametrix for A and $w \in \mathcal{D}_{\max}(A)$. Then

$$w = BAw + Rw, \quad R \in \Psi^{-\infty}(M; E),$$

shows that w is a sum of an section in the image of B and a smooth section, and by (5.10) both of these are in $\mathcal{D}_{\min}(A)$. □

5.5 Sobolev spaces

The second part of Proposition 34 strongly suggests that the L^2-domain of an elliptic pseudodifferential operator of order r is independent of the actual operator involved. This is true, and the common domain is known as the **Sobolev space of order** r. We define these spaces, for any $s \in \mathbb{R}$, by

$$H^s(M; E) = \{u \in \mathcal{C}^{-\infty}(M; E) : Au \in L^2(M; E) \text{ for all } A \in \Psi^s(M; E)\}.$$

We can think of H^s as the sections 'whose first s weak derivatives are in L^2', and if $s \in \mathbb{N}$ this is an accurate description. Notice that

$$s > 0 \implies H^s(M; E) \subseteq L^2(M; E) \subseteq H^{-s}(M; E).$$

It will be useful to have at our disposal a family of operators

$$\begin{aligned} &\mathbb{R} \ni s \mapsto \Lambda_s \in \Psi^s(M; E), \\ &\sigma_s(\Lambda_s) = \mathrm{id}_E \in \mathcal{C}^\infty(\mathbb{S}^*M; \hom(\pi^*E)) \end{aligned} \tag{5.11}$$

that are elliptic, (formally) self-adjoint, strictly positive, and moreover satisfy

$$\Lambda_{-s} = \Lambda_s^{-1} \text{ on } \mathcal{C}^\infty(M; E).$$

These are easy to construct. First define $\Lambda_0 = \mathrm{Id}_E$. Next for $s > 0$, start with any operator $A \in \Psi^{s/2}(M; E)$ with $\sigma_{s/2}(A) = \mathrm{id}_E$ and then define

$$\Lambda_s = A^*A + \mathrm{Id}_E\,.$$

Finally, for $s < 0$, take Λ_{-s} to be the generalized inverse from section 5.3 and note that since Λ_s is (formally) self-adjoint and injective, $\Lambda_{-s} = \Lambda_s^{-1}$. (When there is more than one bundle involved, we will denote Λ_s by Λ_s^E.)

Using these operators, let us define an inner product on $H^s(M; E)$ by

$$\begin{aligned} H^s(M; E) \times H^s(M; E) &\longrightarrow \mathbb{C} \\ (u, v) &\longmapsto \langle u, v\rangle_{H^s(M;E)} = \langle \Lambda_s u, \Lambda_s v\rangle_{L^2(M;E)} \end{aligned} \tag{5.12}$$

and the corresponding norm

$$\|u\|^2_{H^s(M;E)} = \langle u, u\rangle_{H^s(M;E)}.$$

It is easy to see that this inner product gives $H^s(M;E)$ the structure of a Hilbert space.

Theorem 35. *Let M be a compact Riemannian manifold and E, F Hermitian vector bundles over M. A pseudodifferential operator $A \in \Psi^r(M;E,F)$ defines* bounded *operators*

$$A : H^s(M;E) \longrightarrow H^{s-r}(M;F) \text{ for any } s \in \mathbb{R}.$$

For a distribution $u \in \mathcal{C}^{-\infty}(M;E)$, the following are equivalent:
i) $Au \in L^2(M;E)$ for all $A \in \Psi^r(M;E)$, i.e., $u \in H^r(M;E)$
ii) There exists $P \in \Psi^r(M;E,F)$ elliptic such that $Pu \in L^2(M;F)$.
iii) u is in the closure of $\mathcal{C}^\infty(M;E)$ with respect to the norm $\|\cdot\|_{H^r(M;E)}$.

Moreover, every elliptic operator $P \in \Psi^r(M;E,F)$ defines a norm on $H^s(M;E)$,

$$\|u\|_{P,s} = \|Pu\|_{H^{s-r}(M;F)} + \|\mathcal{P}_{\ker P} u\|_{H^s(M;E)}$$

which is equivalent to the norm $\|\cdot\|_{H^s(M;E)}$.

In particular, the Hilbert space structure on $H^s(M;E)$ is independent of the choice of family Λ_s as another choice of family yields equivalent norms.

Proof. Showing that the map $A : \mathcal{C}^\infty(M;E) \subseteq H^s(M;E) \longrightarrow H^{s-r}(M;F)$ is bounded is equivalent to showing that the map

$$\Lambda^F_{s-r} \circ A \circ \Lambda^E_{-s} : \mathcal{C}^\infty(M;E) \subseteq L^2(M;E) \longrightarrow L^2(M;F)$$

is bounded. But this is an operator of order zero, and so is bounded by Proposition 31.

It follows that, for any $A \in \Psi^r(M;E,F)$ and any $s > 0$ there is a constant $C > 0$ such that

$$\|Au\|_{H^{s-r}(M;F)} \leq C\|u\|_{H^s(M;E)}.$$

If $A = P$ is elliptic, and G is the generalized inverse of P, then since it has order $-r$ we have

$$\|Gv\|_{H^{s-r}(M;E)} \leq \|v\|_{H^r(M;F)}.$$

In particular, we have

$$\|Pu\|_{H^{s-r}(M;F)} \geq C\|GPu\|_{H^s(M;E)} = \|u - \mathcal{P}_{\ker P} u\|_{H^s(M;E)}$$

hence

$$\|u\|_{H^s(M;E)} \leq C(\|Pu\|_{H^{s-r}(M;F)} + \|\mathcal{P}_{\ker P} u\|_{H^{s-r}(M;E)}),$$

which proves that $\|u\|_{P,s}$ is equivalent to $\|u\|_{H^s(M;E)}$, since $\ker P$ is a finite dimensional space and all norms on finite dimensional spaces are equivalent. This also establishes the equivalence of (i) and (ii).

For any $s \in \mathbb{R}$, choose $s > r$ and note that $H^s(M;E)$ is the maximal domain of

$$\Lambda_{s-r} : \mathcal{C}^\infty(M;E) \subseteq H^r(M;E) \longrightarrow H^r(M;F)$$

the same argument as in Proposition 34 shows that the minimal domain equals the maximal domain and this shows that $\mathcal{C}^\infty(M;E)$ is dense in $H^s(M;E)$. □

In this way we can refine our understanding of regularity of sections from

$$\mathcal{C}^\infty(M;E) \subseteq L^2(M;E) \subseteq \mathcal{C}^{-\infty}(M;E)$$

to, for any $s > 0$,

$$\mathcal{C}^\infty(M;E) \subseteq H^s(M;E) \subseteq L^2(M;E) \subseteq H^{-s}(M;E) \subseteq \mathcal{C}^{-\infty}(M;E).$$

In fact, the natural L^2-pairing which exhibits $\mathcal{C}^{-\infty}(M;E)$ as the dual of $\mathcal{C}^\infty(M;E)$ and $L^2(M;E)$ as its own dual puts $H^s(M;E)$ in duality with $H^{-s}(M;E)$. Formally, for every $s \in \mathbb{R}$ there is a non-degenerate pairing

$$\begin{array}{ccc} H^s(M;E) \times H^{-s}(M;E) & \longrightarrow & \mathbb{C} \\ (u,v) & \longmapsto & \langle u,v\rangle_{H^s \times H^{-s}} = \langle \Lambda_{-s} u, \Lambda_s v\rangle_{L^2(M;E)} \end{array}$$

and so we can identify $(H^s(M;E))^* = H^{-s}(M;E)$. (Notice that this is different from the pairing (5.12) with respect to which H^s is its own dual.)

Also notice that there are natural inclusions

$$s < t \implies H^t(M:E) \subseteq H^s(M;E)$$

and indeed we can isometrically embed $H^t(M;E)$ into $H^s(M;E)$ by the operator Λ_{s-t}. Notice that since these are pseudodifferential operators of negative order, this inclusion is compact! Put another way, if we have a sequence of distributions in $H^s(M;E)$ that is bounded in $H^t(M;E)$, then it has a convergent sequence in $H^s(M;E)$. This is an L^2-version of the usual Arzela-Ascoli theorem on a compact set K: a sequence of functions in $\mathcal{C}^0(K)$ that is bounded in $\mathcal{C}^1(K)$ has a convergent sequence in $\mathcal{C}^0(K)$.

It is easy to see that for any $\ell \in \mathbb{N}_0$

$$\mathcal{C}^\ell(M;E) \subseteq H^s(M;E) \text{ whenever } s > \ell$$

interestingly there is a sort of converse embedding, known as the Sobolev embedding theorem, (see exercises 9 and 10)

$$H^s(M;E) \subseteq \mathcal{C}^\ell(M;E) \text{ if } \ell \in \mathbb{N}_0, \quad \ell > s - \tfrac{1}{2}\dim M.$$

This lets us see that

$$\mathcal{C}^\infty(M;E) = \bigcap_{s\in\mathbb{R}} H^s(M;E), \text{ and } \mathcal{C}^{-\infty}(M;E) = \bigcup_{s\in\mathbb{R}} H^s(M;E)$$

so that the *scale* of Sobolev spaces $\{H^s(M;E)\}_{s\in\mathbb{R}}$ is a complete refinement of the inclusion of smooth sections into distributional sections.

With the machinery we have built so far, it is a simple matter to refine Corollary 25 to Sobolev spaces.

Theorem 36. *Let M be a compact Riemannian manifold, E and F Hermitian vector bundles over M. Let $P \in \Psi^r(M;E,F)$ be an elliptic operator of order $r \in \mathbb{R}$. We have a refined elliptic regularity statement:*

$$u \in \mathcal{C}^{-\infty}(M;E), Pu \in H^t(M;F) \implies u \in H^{t+r}(M;E) \textit{ for any } t \in \mathbb{R}.$$

Also, for any $s \in \mathbb{R}$, P defines a Fredholm operator either as a closed unbounded operator

$$P : H^{s+r}(M;E) \subseteq H^s(M;E) \longrightarrow H^s(M;F)$$

or as a bounded operator

$$P : H^{s+r}(M;E) \longrightarrow H^s(M;F)$$

and its index is independent of s. Explicitly this means that:
i) The null spaces of P and P^* in either case are finite dimensional (in fact these spaces are themselves independent of s.)
ii) We have an orthogonal decomposition

$$H^s(M;F) = P(H^{s+r}(M;E)) \oplus \ker P^*.$$

A consequence of this theorem is a refined version of the Hodge decomposition. Namely, for any $s \in \mathbb{R}$,

$$H^s(M;\Lambda^kT^*M) = d\left(H^{s+1}(M;\Lambda^{k-1}T^*M)\right) \oplus \delta\left(H^{s+1}(M;\Lambda^{k+1}T^*M)\right) \oplus \mathcal{H}^k(M).$$

5.6 Exercises

Exercise 1.
Make sure you understand how to prove all parts of Theorem 21.

Exercise 2.
Let $A \in \Psi^r(M; E, F)$ be an elliptic pseudodifferential operator. Show that if $B_L, B_R \in \Psi^{-r}(M; F, E)$ satisfy $B_L A - \mathrm{Id}_E \in \Psi^{-\infty}(M; E)$ and $AB_R - \mathrm{Id}_F \in \Psi^{-\infty}(M; F)$ then $B_L - B_R \in \Psi^{-\infty}(M; F, E)$.

Exercise 3.
Show that any linear operator $A : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is Fredholm and compute its index.

Exercise 4.
(*Schur's Lemma*) Suppose (X, μ) is a measure space and $\mathcal{K}$ is a measurable complex-valued function on $X \times X$ satisfying

$$\int_X |\mathcal{K}(x, y)|\, d\mu(x) \leq C_1, \quad \int_X |\mathcal{K}(x, y)|\, d\mu(y) \leq C_2,$$

for all y and for all x respectively. Show that the prescription

$$Ku(x) = \int_X \mathcal{K}(x, y)u(y)\, d\mu(y)$$

defines a bounded operator on $L^p(X, \mu)$ for each $p \in (1, \infty)$ with operator norm

$$\|K\|_{L^p \to L^p} \leq C_1^{1/p} C_2^{1/q}, \text{ where } q \text{ satisfies } \tfrac{1}{p} + \tfrac{1}{q} = 1.$$

Hint:It suffices to bound $|\langle Ku, v\rangle|$ for u and v with $\|u\|_{L^p} \leq 1$, $\|v\|_{L^q} \leq 1$. Start with the expression

$$|\langle Ku, v\rangle| = \left| \int_X \int_X \mathcal{K}(x, y)u(y)v(x)\, d\mu(y) d\mu(x) \right|$$

and apply the inequality

$$ab \leq \tfrac{a^p}{p} + \tfrac{b^q}{q}$$

with $a = tu$ and $b = t^{-1}v$ where t is a positive real number to be determined. Minimize over t to get the desired bound.

Exercise 5.
Show that the linear operator $T : \mathcal{C}^\infty([0, 1]) \subseteq L^2([0, 1]) \longrightarrow L^2([0, 1])$ given by $Tu = \partial_x u$ is not bounded and not closed.

Exercise 6.
Let $\mathcal{H}_1 = L^2(\mathbb{R})$, $\mathcal{H}_2 = \mathbb{R}$ and consider the operator

$$Au = \int_{\mathbb{R}} u(x)\, dx$$

from $\mathcal{H}_1$ to $\mathcal{H}_2$ with domain $\mathcal{D}(A) = \mathcal{C}_c^\infty(\mathbb{R})$. Prove that the closure of the graph of A is *not* the graph of a linear operator.
Hint: For each $n \in \mathbb{N}$, let

$$u_n(x) = \begin{cases} \frac{1}{n} & \text{if } |x| \leq n \\ 0 & \text{if } |x| > n \end{cases}$$

Show that $u_n \to 0$ in $L^2(\mathbb{R})$ and $Au_n = 2$ for all n. Why does this solve the exercise?

Exercise 7.
Let A be a linear operator $A : \mathcal{D}(A) \subseteq \mathcal{H} \longrightarrow \mathcal{H}$. Given $\lambda \in \mathbb{C}$, show that $A - \lambda\,\mathrm{Id}$ is closed if and only if A is closed.

Exercise 8.
Show that the operator $A = \partial_x \sin(x) \partial_x$ on the circle is symmetric, and that $\mathcal{D}_{\min}(A) \neq \mathcal{D}_{\max}(A)$.

Exercise 9.
One can also define Sobolev spaces on $\mathbb{R}^m$ where there is a natural choice for the operators Λ_s in (5.11), namely

$$\Lambda_s f(x) = \mathcal{F}^{-1}\left((1 + |\xi|^2)^{s/2} \mathcal{F}(f)\right)$$

and then as before $H^s(\mathbb{R}^m)$ is the closure of $\mathcal{C}_c^\infty(\mathbb{R}^m)$ with respect to the norm $\|f\|_s = \|\Lambda_s f\|_{L^2(\mathbb{R}^m)}$.
Let $s \in \mathbb{R}$ and $k \in \mathbb{N}_0$ such that $k < s - \frac{1}{2}m$ show that there is a constant C such that

$$\sup_{|\alpha| \leq k} \sup_{\zeta \in \mathbb{R}^m} |D^\alpha f| \leq C \|f\|_s.$$

Hint: First note that $\|\phi\|_{L^\infty(\mathbb{R}^m)} \leq \|\mathcal{F}(\phi)\|_{L^1(\mathbb{R}^m)}$ for any $\phi \in \mathcal{S}(\mathbb{R}^m)$. Next note that

$$\|\mathcal{F}(D^\alpha f)\|_{L^1(\mathbb{R}^m)} = \|\xi^\alpha \mathcal{F}(f)\|_{L^1(\mathbb{R}^m)} \leq \|(1 + |\xi|^2)^{|\alpha|/2} \mathcal{F}(f)\|_{L^1(\mathbb{R}^m)}$$

Finally, write $(1 + |\xi|^2)^{k/2} = (1 + |\xi|^2)^{s/2}(1 + |\xi|^2)^{(k-s)/2}$ and use Cauchy-Schwarz.

Exercise 10.
Prove the Sobolev embedding theorem for integer degree Sobolev spaces: If $s \in \mathbb{N}_0$ there is a continuous embedding of $H^s(M; E)$ into $\mathcal{C}^\ell(M; E)$ for any $\ell \in \mathbb{N}_0$ satisfying $\ell < s - \frac{1}{2}\dim M$.

Hint: Let $f \in H^s(M;E)$, use the fact that multiplication by a smooth function is a pseudodifferential operator of order zero to reduce to the case where f is supported in a coordinate chart $\phi : \mathcal{U} \subseteq \mathbb{R}^m \longrightarrow M$ that trivializes E. Since s is a natural number the Sobolev norm on $H^s(M;E)$ is equivalent to a norm $\|\cdot\|_D$ with D a differential operator, use this to show that $f \circ \phi \in H^s(\mathcal{U};\mathbb{C}^{\operatorname{rank} E})$. Use the previous exercise to conclude that f has ℓ continuous derivatives.
One can use a technique called 'interpolation' to deduce the general Sobolev embedding theorem from this integer degree version.

Exercise 11.
(The adjoint of a linear operator) Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be $\mathbb{C}$-Hilbert spaces and $A : \mathcal{D}(A) \subseteq \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ a linear operator. The (Hilbert space) **adjoint** of A, is a linear operator from $\mathcal{H}_2$ to $\mathcal{H}_1$ defined on

$$\mathcal{D}(A)^* = \{v \in \mathcal{H}_2 : \exists w \in \mathcal{H}_1 \text{ s.t. } \langle Au, v\rangle_{\mathcal{H}_2} = \langle u, w\rangle_{\mathcal{H}_1} \text{ for all } u \in \mathcal{D}(A)\}$$

by the prescription

$$\langle Au, v\rangle = \langle u, A^*v\rangle \text{ for all } u \in \mathcal{D}(A), v \in (\mathcal{D}(A))^*.$$

(Note that we always assume our linear operators have dense domains, so A^*v is determined by this relation.) If $\mathcal{H}_1 = \mathcal{H}_2$, a linear operator $(A, \mathcal{D}(A))$ is **self-adjoint** if $\mathcal{D}(A)^* = \mathcal{D}(A)$ and

$$\langle Au, v\rangle_{\mathcal{H}_1} = \langle u, Av\rangle_{\mathcal{H}_1} \text{ for all } u, v \in \mathcal{D}(A).$$

i) Show that the graph of $(A^*, \mathcal{D}(A)^*)$ is the orthogonal complement in $\mathcal{H}_2 \times \mathcal{H}_1$ of the set

$$\{(-Au, u) \in \mathcal{H}_2 \times \mathcal{H}_1 : u \in \mathcal{D}(A)\}$$

so in particular $(A^*, \mathcal{D}(A)^*)$ is always a closed operator (even if A is not a closed operator).
ii) If $(A, \mathcal{D}(A))$ is a closed operator, show that $(\mathcal{D}(A))^*$ is a dense subspace of $\mathcal{H}_2$ (e.g., by showing that $z \perp \mathcal{D}(A)^* \implies z = 0$) so we can define the adjoint of $(A^*, \mathcal{D}(A)^*)$. Prove that $((A^*)^*, (\mathcal{D}(A)^*)^*) = (A, \mathcal{D}(A))$.
ii) Suppose $\mathcal{H}_1 = L^2(M;E)$, $\mathcal{H}_2 = L^2(M;F)$ where M is a Riemannian manifold and E, F are Hermitian vector bundles, and $A \in \Psi^r(M;E,F)$ for some $r \in \mathbb{R}$. Let $A^\dagger$ denote the *formal* adjoint of A, show that the Hilbert space adjoint of the linear operator

$$A : \mathcal{C}^\infty(M;E) \subseteq L^2(M;E) \longrightarrow L^2(M;F)$$

is the linear operator

$$A^\dagger : \mathcal{D}_{\max}(A^\dagger) \subseteq L^2(M;F) \longrightarrow L^2(M;E).$$

Hint: Recall the definition of the action of $A^\dagger$ on distributions.
iii) Let $\mathcal{H}_1 = \mathcal{H}_2 = L^2(M;E)$. Show that if $A \in \Psi^r(M;E)$ is elliptic and formally self-adjoint, then the unique closed extension of A as an operator on $L^2(M;E)$ is self-adjoint.

Exercise 12.
Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be $\mathbb{C}$-Hilbert spaces and $A : \mathcal{D}(A) \subseteq \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ a linear operator. Recall that A is Fredholm if:
i) $\ker A$ is a finite dimensional subspace of $\mathcal{H}_1$
ii) $\ker A^*$ is a finite dimensional subspace of $\mathcal{H}_2$
iii) $\mathcal{H}_2$ has an orthogonal decomposition $\mathcal{H}_2 = \mathrm{Im}(A) \oplus \ker A^*$ (i.e., $\mathrm{Im}(A)$ is closed).
Prove that an operator is Fredholm if and only if it has an inverse modulo compact operators. That is, there exists $B : \mathcal{D}(B) \subseteq \mathcal{H}_2 \longrightarrow \mathcal{H}_1$ such that $AB - \mathrm{Id}_{\mathcal{H}_2}$ and $BA - \mathrm{Id}_{\mathcal{H}_1}$ are compact operators.
Hint: One can follow the arguments above very closely, replacing smoothing operators by compact operators.

5.7 Bibliography

Same as Lecture 4.

Lecture 6

Spectrum of the Laplacian

6.1 Spectrum and resolvent

Let $\mathcal{H}$ be a complex Hilbert space and $A : \mathcal{D}(A) \subseteq \mathcal{H} \longrightarrow \mathcal{H}$ a linear operator. We divide the complex numbers into the **resolvent set of** A

$$\rho(A) = \{\lambda \in \mathbb{C} : A - \lambda \operatorname{Id}_{\mathcal{H}} \text{ has a bounded inverse}\}$$

and its complement, the **spectrum of** A,

$$\operatorname{Spec}(A) = \mathbb{C} \setminus \rho(A).$$

The **resolvent** of A is the family of bounded linear operators

$$\begin{aligned} \rho(A) &\longrightarrow \mathcal{B}(\mathcal{H}) \\ \lambda &\longmapsto R(\lambda) = (A - \lambda \operatorname{Id}_{\mathcal{H}})^{-1} \end{aligned}$$

This is, in a natural sense, a holomorphic family of bounded operators[1]. Notice that the graph of the resolvent satisfies

$$\{(u, R(\lambda)u) : u \in \mathcal{H}\} = \{((A - \lambda \operatorname{Id}_{\mathcal{H}})v, v) : v \in \mathcal{H}\}$$

and hence $\lambda \in \rho(A)$ implies that the operator $A - \lambda \operatorname{Id}_{\mathcal{H}}$ is closed, and so by exercise 7 of the previous lecture, that A is closed. In other words if A is not closed then $\operatorname{Spec}(A) = \mathbb{C}$.

[1] See for instance, Rudin Functional Analysis, for the meaning of a holomorphic vector-valued function.

Since a complex number λ is in the spectrum of A if $A - \lambda\,\mathrm{Id}_{\mathcal{H}}$ fails to have a bounded inverse, there is a natural division of Spec(A) by taking into account *why* there is no bounded inverse: the **point spectrum** or **eigenvalues** of A consists of

$$\mathrm{Spec}_{pt}(A) = \{\lambda \in \mathbb{C} : A - \lambda\,\mathrm{Id}_{\mathcal{H}} \text{ is not injective }\},$$

the **continuous spectrum** of A consists of

$$\mathrm{Spec}_{c}(A) = \{\lambda \in \mathbb{C} : A - \lambda\,\mathrm{Id}_{\mathcal{H}} \text{ is injective and has dense range, but is not surjective}\},$$

everything else is called the **residual spectrum,**

$$\mathrm{Spec}_{res}(A) = \{\lambda \in \mathbb{C} : A - \lambda\,\mathrm{Id}_{\mathcal{H}} \text{ is injective but its range is not dense}\}.$$

If $\lambda \in \mathrm{Spec}_{pt}(A)$ then $\ker(A - \lambda\,\mathrm{Id}_{\mathcal{H}})$ is known as the **λ-eigenspace** and will be denoted $E_\lambda(A)$, its elements are called **eigenvectors**.

Recall that the adjoint of a bounded operator $A \in \mathcal{B}(\mathcal{H})$ is the unique operator $A^* \in \mathcal{B}(\mathcal{H})$ satisfying

$$\langle Au, v\rangle_{\mathcal{H}} = \langle u, A^* v\rangle_{\mathcal{H}} \text{ for all } u, v \in \mathcal{H}.$$

Just as on finite dimensional spaces, for any operator eigenvectors of different eigenvalues are always linearly independent and if the operator is self-adjoint then they are also orthogonal (and the eigenvalues are real numbers).

Let us recall some other elementary functional analytic facts.

Proposition 37. *Let $\mathcal{H}$ be a complex Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. If $T = T^*$ then*

$$\|T\|_{\mathcal{H}\to\mathcal{H}} = \sup_{\|u\|=1} |\langle Tu, u\rangle_{\mathcal{H}}| = \max_{\lambda\in\mathrm{Spec}(T)} |\lambda|.$$

Proof. Let us abbreviate $|T| = \sup_{\|u\|=1} |\langle Tu, u\rangle_{\mathcal{H}}|$. Cauchy-Schwarz shows that $|T| \leq \|T\|_{\mathcal{H}\to\mathcal{H}}$. To prove the opposite inequality, let $u, v \in \mathcal{H}$ have length one and assume that $\langle Tu, v\rangle_{\mathcal{H}} \in \mathbb{R}$ (otherwise we just multiply v by a unit complex number to make this real). We have

$$\begin{aligned} 4\langle Tu, v\rangle_{\mathcal{H}} &= \langle T(u+v), (u+v)\rangle_{\mathcal{H}} - \langle T(u-v), (u-v)\rangle_{\mathcal{H}} \\ &\leq |T|(\|u+v\|_{\mathcal{H}}^2 + \|u-v\|_{\mathcal{H}}^2) = 2|T|(\|u\|_{\mathcal{H}}^2 + \|v\|_{\mathcal{H}}^2) = 4|T| \end{aligned}$$

and hence

$$\|T\|_{\mathcal{H}\to\mathcal{H}} = \sup_{\|u\|_{\mathcal{H}}=1} \|Tu\|_{\mathcal{H}} = \sup_{\|u\|_{\mathcal{H}}=1} \sup_{\|v\|_{\mathcal{H}}=1} \langle Tu, v\rangle_{\mathcal{H}} \leq |T|.$$

Next, replacing T by $-T$ if necessary, we can assume that $\sup_{\|u\|_{\mathcal{H}}=1}\langle Tu, u\rangle_{\mathcal{H}} > 0$. Choose a sequence of unit vectors (u_n) such that

$$\lim\langle Tu_n, u_n\rangle = |T|$$

and note that

$$\|(T - |T|\,\mathrm{Id})u_n\|_{\mathcal{H}}^2 = \|Tu_n\|_{\mathcal{H}}^2 + |T|^2 - 2|T|\langle Tu_n, u_n\rangle \leq 2|T|^2 - 2|T|\langle Tu_n, u_n\rangle \to 0$$

implies that $(T - |T|\,\mathrm{Id})$ is not invertible, so $|T| \in \mathrm{Spec}(T)$.

Finally, we need to show that if $\lambda > \|T\|_{\mathcal{H}\to\mathcal{H}}$ then $\lambda \in \rho(T)$, i.e., that $(T - \lambda\,\mathrm{Id}_{\mathcal{H}})$ is invertible. It suffices to note that $A = \frac{1}{\lambda}T$ has operator norm less than one and hence the series

$$\sum_{k\geq 0}(-1)^k A^k$$

converges in $\mathcal{B}(\mathcal{H})$ to an inverse of $\mathrm{Id}_{\mathcal{H}} - A$. □

With those preliminaries recalled, we can now establish the spectral theorem for compact operators.

Theorem 38. *Let $\mathcal{H}$ be a complex Hilbert space and $K \in \mathcal{K}(\mathcal{H})$ a compact operator. The spectrum of K satisfies*

$$\mathrm{Spec}(K) \subseteq \mathrm{Spec}_{pt}(K) \cup \{0\}$$

with equality if $\mathcal{H}$ is infinite dimensional, in which case zero may be an eigenvalue or an element of the continuous spectrum. Each non-zero eigenspace of K is finite dimensional and the point spectrum is either finite or consists of a sequence converging to $0 \in \mathbb{C}$.

If K is also self-adjoint there is a complete orthonormal eigenbasis of $\mathcal{H}$ and we can expand K in the convergent series

$$K = \sum_{\lambda\in\mathrm{Spec}(K)} \lambda\mathcal{P}_\lambda$$

where $\mathcal{P}_\lambda$ is the orthogonal projection onto $E_\lambda(K)$.

Proof. First notice that if a compact operator on a space V is invertible, then the unit ball of V is compact and hence V is finite dimensional. Thus if $\mathcal{H}$ is infinite dimensional, then zero must be in the spectrum of K.

Let $\lambda \neq 0$ then $K - \lambda\,\mathrm{Id}_{\mathcal{H}} = -\lambda(\mathrm{Id}_{\mathcal{H}} - \frac{1}{\lambda}K)$ and we know that $\mathrm{Id}_{\mathcal{H}} - \frac{1}{\lambda}K$ is a Fredholm operator of index zero (since we can connect it to $\mathrm{Id}_{\mathcal{H}}$ along Fredholm operators). Thus its null space is finite dimensional and if it is injective it is surjective so by the closed graph theorem has a bounded inverse. This shows that $\mathrm{Spec}(K)\setminus\{0\}$ consists of eigenvalues with finite dimensional eigenspaces.

Notice that, for any $\varepsilon > 0$ the operator K restricted to

$$\mathrm{span}\left(\bigcup_{\substack{\lambda\in\mathrm{Spec}(K)\\ |\lambda|>\varepsilon}} E_\lambda(K)\right)$$

is both compact and invertible, hence this space must be finite dimensional. Thus if the spectrum is not finite, the eigenvalues form a sequence converging to zero.

Finally, if K is self-adjoint, let $\mathcal{H}'$ be the orthogonal complement of

$$\mathrm{span}\left(\bigcup_{\lambda\in\mathrm{Spec}_{pt}(K)} E_\lambda(K)\right).$$

Notice that K preserves $\mathcal{H}'$, since if $v \in E_\lambda(K)$ and $w \in \mathcal{H}'$ we have

$$\langle v, Kw\rangle_{\mathcal{H}} = \langle Kv, w\rangle_{\mathcal{H}} = \lambda\langle v, w\rangle = 0.$$

Therefore $K\big|_{\mathcal{H}'}$ is a compact self-adjoint operator with

$$\mathrm{Spec}(K\big|_{\mathcal{H}'}) = \{0\}$$

and so must be the zero operator. But this means that $\mathcal{H}'$ is both in the null space of K and orthogonal to the null space of K, so $\mathcal{H}' = \{0\}$. Similarly, notice that

$$K - \sum_{\substack{\lambda\in\mathrm{Spec}(K)\\ |\lambda|>\varepsilon}} \lambda\mathcal{P}_\lambda$$

is a self-adjoint operator with spectrum contained in $[-\varepsilon, \varepsilon] \subseteq \mathbb{R}$. It follows that its operator norm is at most ε which establishes the convergence of the series to K. □

This immediately yields the spectral theorem for elliptic operators.

Theorem 39. *Let M be a closed manifold, E a Hermitian vector bundle over M, and $P \in \Psi^r(M; E)$ an elliptic operator of positive order $r \in \mathbb{R}$. If $\rho(P) \neq \emptyset$ then* $\mathrm{Spec}(P)$ *consists of a sequence of eigenvalues diverging to infinity, with each eigenspace a finite dimensional space of smooth sections of E.*

If P is self-adjoint then $\rho(P) \neq \emptyset$, the eigenvalues are real, the eigenspaces are orthogonal, and there is an orthonormal eigenbasis of $L^2(M; E)$.

Proof. If $\zeta \in \rho(P)$ then $(P - \zeta \operatorname{Id}_E)^{-1} \in \Psi^{-r}(M;E)$ is a compact operator on $L^2(M;E)$ and we can apply the spectral theorem for compact operators. So it suffices to note that

$$Pv = \lambda v \iff (P - \zeta \operatorname{Id}_E)^{-1} v = (\lambda - \zeta)^{-1} v.$$

If P is self-adjoint then $\operatorname{Spec}(P) \subseteq \mathbb{R}$, so $\rho(P) \neq \emptyset$. □

This is the case in particular for the Laplacian of a Riemannian metric. A classical problem is to understand how much of the geometry can be recovered from the spectrum of its Laplacian. This was popularized by an article of Mark Kac entitled 'Can you hear the shape of a drum?'.

6.2 The heat kernel

The fundamental solution of the heat equation

$$\begin{cases} (\partial_t + \Delta) u = 0 \\ u\big|_{t=0} = u_0 \end{cases}$$

is a surprisingly useful tool in geometric analysis. In particular, it will help us to show that the spectrum of the Laplacian contains lots of geometric information. We will discuss an approach to the heat equation known as the Hadamard parametrix construction, following Richard Melrose.

For simplicity we will work with the scalar Laplacian, but the construction extends to any Laplace-type operator with only notational changes.

On any compact Riemannian manifold, the **heat operator**

$$e^{-t\Delta} : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^+ \times M)$$

is the unique linear map satisfying

$$\begin{cases} (\partial_t + \Delta) e^{-t\Delta} u_0 = 0 \\ \left(e^{-t\Delta} u_0\right)\big|_{t=0} = u_0 \end{cases}$$

Its integral kernel $H(t,x,y)$ is known as the **heat kernel** and it is *a priori* a distribution on $\mathbb{R}^+ \times M \times M$, but we will soon show that it is a smooth function for $t > 0$ and understand its regularity as $t \to 0$. Once we know $H(t,x,y)$ then we can solve the heat equation, even with an inhomogeneous term, since

$$\begin{cases} (\partial_t + \Delta)v = f(t,x) \\ v\big|_{t=0} = v_0(x) \end{cases} \implies$$

$$v(t,x) = \int_M H(t,x,y)v_0(y)\ \mathrm{dvol}_g(y) + \int_0^t \int_M H(t-s,x,y)f(s,y)\ \mathrm{dvol}_g(y)ds.$$

It is straightforward to write $e^{-t\Delta}$ as an operator on $L^2(M)$ by using the spectral data of Δ. Indeed, if $\{\phi_j\}$ is an orthonormal basis of $L^2(M)$, with $\Delta\phi_j = \lambda_j\phi_j$, then we have

$$H(t,x,y) = \sum_j e^{-t\lambda_j}\phi_j(x)\overline{\phi_j(y)} \text{ for all } t > 0.$$

However it is hard to extract geometric information from this presentation.

On Euclidean space, the heat kernel is well-known (see exercise 7)

$$H_{\mathbb{R}^m}(t,x,y) = \frac{1}{(4\pi t)^{m/2}} \exp\left(-\frac{|x-y|^2}{4t}\right).$$

Notice that $H_{\mathbb{R}^m}$ is smooth in t,x,y as long as $t > 0$. As $t \to 0$, $H_{\mathbb{R}^m}$ vanishes exponentially fast away from $\mathrm{diag}_{\mathbb{R}^m} = \{x = y\}$ and blows-up as we approach

$$\{0\} \times \mathrm{diag}_{\mathbb{R}^m}.$$

Notice that $t^{m/2}H_{\mathbb{R}^m}$ is still singular as we approach this submanifold since the argument of the exponential as we approach a point $(0,x,x)$ depends not just on x, but also *on the way we approach*. For instance, for any $z \in \mathbb{R}^m$, the limit of $t^{m/2}H(t,x,y)$ as $t \to 0$ along the curve $(t, x + t^{1/2}z, x)$ is

$$\exp\left(-\frac{|z|}{4}\right).$$

To understand the singularity at $\{0\} \times \mathrm{diag}_{\mathbb{R}^m}$ we need to take into account the different ways of approaching this submanifold. We do this by introducing 'parabolic' polar coordinates (parabolic because the t-direction should scale differently from the $x - y$ direction). Define

$$\mathbb{S}_{\mathcal{P}} = \{(\theta,\omega) \in \mathbb{R}^+ \times \mathbb{R}^m : \theta + |\omega|^2 = 1\}$$

and notice that any point $(t,x,y) \in \mathbb{R}^+ \times \mathbb{R}^m \times \mathbb{R}^m$ can be written

$$t = r^2\theta, \quad x = x, \quad y = x - r\omega, \text{ with } r = \sqrt{t + |x-y|^2}.$$

Indeed, (r,θ,ω,x) are the parabolic polar coordinates for $\mathbb{R}^+ \times \mathbb{R}^m \times \mathbb{R}^m$. In these coordinates, we can write $t^{m/2} H_{\mathbb{R}^m}$ as

$$\exp\left(-\frac{|\omega|^2}{4\theta}\right)$$

and, since θ and ω never vanish simultaneously, this is now a smooth function! We say that these coordinates have *resolved the singularity* of the heat kernel.

Taking a closer look at the parabolic polar coordinates, notice that these are coordinates only when $r \neq 0$. Each point in the diagonal of $\mathbb{R}^m$ at $t = 0$ corresponds to $r = 0$ and to a whole $\mathbb{S}_{\mathcal{P}}$. A better way of thinking about this is that we have constructed a new manifold with corners (the heat space of $\mathbb{R}^m$)

$$H\mathbb{R}^m = \mathbb{R}^+ \times \mathbb{S}_{\mathcal{P}} \times \mathbb{R}^m$$

with a natural map

$$\begin{aligned} H\mathbb{R}^m &\xrightarrow{\beta} \mathbb{R}^+ \times \mathbb{R}^m \times \mathbb{R}^m \\ (r,\theta,\omega,x) &\longmapsto (r^2\theta, x, y - r\omega) \end{aligned}$$

which restricts to a diffeomorphism

$$H\mathbb{R}^m \setminus \{r = 0\} \longrightarrow (\mathbb{R}^+ \times \mathbb{R}^m \times \mathbb{R}^m) \setminus (\{0\} \times \operatorname{diag}_M).$$

We refer to $H\mathbb{R}^m$ (together with the map β) as the *parabolic blow-up of* $\mathbb{R}^+ \times \mathbb{R}^m \times \mathbb{R}^m$ *along* $\{0\} \times \operatorname{diag}_M$.

Now let (M, g) be a Riemannian manifold. As a first approximation to the heat kernel of Δ_g consider

$$G(t,x,y) = \frac{1}{(4\pi t)^{m/2}} \exp\left(-\frac{d(x,y)^2}{4t}\right).$$

Analogously to the discussion above, this is a smooth function on $\mathbb{R}^+ \times M^2$ except at $\{0\} \times \operatorname{diag}_M$, and we will resolve this singularity by means of a 'parabolic blow-up'. What should replace the diagonal at time zero? Consider the space $\mathbb{R}^+ \times TM$ with its natural map down to M,

$$\pi : \mathbb{R}^+ \times TM \longrightarrow TM \longrightarrow M.$$

At each point $p \in M$ define

$$\mathbb{S}_{\mathcal{P}}(p) = \{(\theta,\omega) \in \mathbb{R}^+ \times T_pM : t + |\omega|_g^2 = 1\}$$

and then

$$\mathbb{S}_{\mathcal{P}}(M) = \bigcup_{p\in M} \mathbb{S}_{\mathcal{P}}(p).$$

The map π restricts to a map $\pi : \mathbb{S}_{\mathcal{P}}(M) \longrightarrow M$ which is a locally trivial fibration with fibers $\mathbb{S}_{\mathcal{P}}$. Let us define the heat space of M as the disjoint union

$$HM = (\mathbb{R}^+ \times M^2 \setminus \{0\} \times \mathrm{diag}_M) \bigsqcup \mathbb{S}_{\mathcal{P}} M,$$

endowed with a smooth structure as a manifold with corners[2] where we attach $\mathbb{S}_{\mathcal{P}}M$ to $\mathbb{R}^+ \times M^2 \setminus \{0\} \times \mathrm{diag}_M$ by the exponential map of g. There is a natural 'blow-down map'

$$\beta : HM \longrightarrow \mathbb{R}^+ \times M^2$$

which is the identity on $\mathbb{R}^+ \times M^2 \setminus \{0\} \times \mathrm{diag}_M$ and is equal to π on $\mathbb{S}_{\mathcal{P}}M$.

This means that given a normal coordinate system for $\mathbb{R}^+ \times M^2$ at $(0, p, p)$,

$$\mathcal{U} \subseteq \mathbb{R}^+ \times T_pM \times T_pM \longrightarrow \mathbb{R}^+ \times M^2$$

we get a coordinate system around $\beta^{-1}(0, p, p)$ by introducing parabolic polar coordinates as before, but now on the fibers of T_pM. Thus, starting with a coordinate system induced by $\exp_p : T_pM \longrightarrow M$, say $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_m)$, we get a coordinate system on HM,

$$r, (\theta, \omega), \zeta \in \mathbb{R}^+ \times \mathbb{S}_{\mathcal{P}} \times \mathbb{R}^m$$

with the identification of $r = 0$ with $\mathbb{S}_{\mathcal{P}}(\exp_p(\zeta))$ and the identification of points with $r \neq 0$ with

$$(r^2\theta, \exp_p(r\omega + \zeta), \exp_p(\zeta)) \in (\mathbb{R}^+ \times M^2 \setminus \{0\} \times \mathrm{diag}_M).$$

Recall that the distance between $\exp_p(r\omega + \zeta)$ and $\exp_p(\zeta)$ is given by $|r\omega|^2_{g(p)} + \mathcal{O}(|r\omega|^4_{g(p)})$ so that our approximation to the heat kernel in these coordinates is

$$\begin{aligned}
&\frac{1}{(4\pi r^2\theta)^{m/2}} \exp\left(-\frac{|r\omega|^2_{g(p)} + \mathcal{O}(|r\omega|^4_{g(p)})}{4r^2\theta} \right) \\
&= \frac{1}{(4\pi r^2\theta)^{m/2}} \exp\left(-\frac{|\omega|^2_{g(p)}}{4\theta} + \mathcal{O}\left(\frac{r^2|\omega|^4_{g(p)}}{\theta} \right) \right)
\end{aligned}$$

and this is r^{-m} times a smooth function, so we have again resolved the singularity.

This computation shows that G approximates the Euclidean heat kernel to leading order as $t \to 0$. That is enough to see that, for any $u_0 \in \mathcal{C}^\infty(M)$,

$$\int_M G(t, x, y)\chi(d_g(x, y))u_0(y)\ \mathrm{dvol}_g(y) \to u_0(x)$$

[2] For a careful proof, see Proposition 7.7 in Richard B. Melrose, The Atiyah-Patodi-Singer index theorem. Research Notes in Mathematics, 4. A K Peters, Ltd., Wellesley, MA, 1993.

where $\chi(d_g(x,y))$ is a cut-off function near the diagonal . In fact, we will proceed much as in the construction of an elliptic parametrix and find a sequence of functions A_j such that the operator

$$\widetilde{H}(r,(\theta,\omega),x) \sim \beta^*(\chi(d_g(x,y))G(t,x,y)) \sum_{j\geq 0} t^{j/2} A_j(r,(\theta,\omega),x),$$

satisfies $\beta^*(\partial_t+\Delta)\widetilde{H}(r,(\theta,\omega),x) = \mathcal{O}(r^\infty)$ as $r \to 0$. Here we are using the fact that, since β is the identity away from the blown-up face, it makes sense to pull-back the differential operator $\partial_t+\Delta$; in a moment, we will compute this in local coordinates.

Let us denote $\beta^*(\chi(d_g(x,y))G(t,x,y))$ by $\widetilde{G}(r,(\theta,\omega),x)$. We will assume inductively that we have found $A_0, A_1, \ldots, A_N$ such that

$$\beta^*(\partial_t+\Delta)\left(\widetilde{G}(r,(\theta,\omega),x)\sum_{j\geq 0}^{N} r^j A_j(r,(\theta,\omega),x)\right) = r^{-n+N+1}E_N(r,(\theta,\omega),x)$$

with E_N a smooth function on the blown-up space, vanishing to infinite order at $t=0$ away from the diagonal, and then we need to find $A_{N+1}(r,(\theta,\omega),x)$ such that

$$r^{-n+N+1}E_N(r,(\theta,\omega),x) + \beta^*(\partial_t+\Delta)\left(r^{(N+1)}\widetilde{G}A_{N+1}\right)$$

vanishes to order $-n+N+2$ at the blown-up boundary face $\{r=0\}$, i.e.,

$$\left[r^{n-N-1}\beta^*(\partial_t+\Delta)\left(r^{(N+1)}\widetilde{G}A_{N+1}\right)\right]_{r=0} = -E_N(0,(\theta,\omega),x).$$

To carry out this construction, it is convenient to use a different set of coordinates, 'projective coordinates'

$$\tau=\sqrt{t}, \quad Y=\frac{\zeta}{\sqrt{t}}, \quad x$$

in which we can compute

$$\beta^*((t\partial_t)f(t,x,y)) = \left(\tfrac{1}{2}\tau\partial_\tau - \tfrac{1}{2}\sum Y_j\partial_{Y_j}\right)\beta^*(f(t,x,y))$$
$$\beta^*(\partial_{z_j}f(t,x,y)) = \tfrac{1}{\tau}(\partial_{Y_j}+\tau V_j)\beta^*(f(t,x,y))$$

for some vector fields V_j on HM. In these coordinates,

$$\beta^*G(t,x,y) = \frac{1}{(4\pi)^{m/2}\tau^m}\exp\left(-\frac{|Y|}{4}+\mathcal{O}(\tau^2|Y|^2)\right)$$

so altogether we have to solve

$$\left(\Delta_Y - \tfrac{1}{2}Y\cdot\partial_Y + \frac{N+1-m}{2}\right)\left(e^{-|Y|^2/4}A_{N+1}\right) = -E_N(0,Y,x).$$

and taking the Fourier transform in Y, this becomes

$$(|\xi|^2 + \tfrac{1}{2}\xi\cdot\partial_\xi + C)(f(\xi)) = h(\xi) \xrightarrow{\rho=|\eta|} (\rho\partial_\rho + 2\rho^2 + C)(f(\xi)) = h(\xi)$$

and one can check that this has a unique solution which decays rapidly as $\rho\to\infty$ and depends smoothly on $\widehat{\xi} = \frac{\xi}{|\xi|}$. Taking the inverse Fourier transform we get the required A_{N+1}.

Having inductively constructed all of the A_j, now let $\widetilde{H}(r,(\theta,\omega),x)$ be a function on HM such that $r^m\widetilde{H}(r,(\theta,\omega),x)$ is smooth on HM, vanishes to infinite order at $t=0$ away from the diagonal, and at the blown-up boundary face has Taylor expansion $G(r,(\theta,\omega),x)\sum r^jA_j$. (The existence of such a function follows from the classical Borel's theorem, much like asymptotic completeness of symbols above.)

This function $\widetilde{H}$ is our parametrix for the heat kernel. It solves the heat equation to infinite order as $r\to 0$, in that

$$S = \beta^*(\partial_t+\Delta)\widetilde{H} \in \bigcap_{n\in\mathbb{N}} r^n\mathcal{C}^\infty(HM) = \bigcap_{n\in\mathbb{N}} t^n\mathcal{C}^\infty(\mathbb{R}^+\times M^2).$$

Next we use it to find the actual solution to the heat equation.

Notice that we can think of a function $F\in\mathcal{C}^\infty(\mathbb{R}^+\times M^2)$ as defining an operator

$$\begin{aligned}\mathcal{C}^\infty(M) &\xrightarrow{F} \mathcal{C}^\infty(\mathbb{R}^+\times M)\\ u &\longmapsto \textstyle\int_M F(t,x,y)u(y)\,dy\end{aligned}$$

but we can also think of it as defining an operator

$$\begin{aligned}\mathcal{C}^\infty(\mathbb{R}^+\times M) &\xrightarrow{\mathrm{Op}(F)} \mathcal{C}^\infty(\mathbb{R}^+\times M)\\ u &\longmapsto \textstyle\int_0^t\int_M F(s,x,y)u(t-s,y)\,dyds\end{aligned}$$

and the advantage is that then it makes sense to compose operators. The composition of the operators corresponding to the functions F_1 and F_2 is easily seen to be an operator of the same form, but now associated to the function

$$F_1 * F_2(t,x,y) = \int_0^t\int_M F_1(s,x,z)F_2(t-s,z,y)\,dzds.$$

With respect to this product the identity is associated to the distribution $\delta(t)\delta(x-y)$ and so we can recognize the heat equation for the heat kernel

$$\begin{cases} (\partial_t + \Delta)H(t,x,y) = 0 & \text{if } t > 0 \\ H(0,x,y) = \delta(x-y) \end{cases}$$

as the equation $\mathrm{Op}((\partial_t + \Delta)H(t,x,y)) = \mathrm{Id}$.

Our parametrix satisfies

$$\mathrm{Op}(\beta^*(\partial_t + \Delta)\widetilde{H}) = \mathrm{Id} + \mathrm{Op}(S)$$

so we need to invert $\mathrm{Id} + \mathrm{Op}(S)$. This can be done explicitly by a series

$$(\mathrm{Id} + \mathrm{Op}(S))^{-1} = \mathrm{Op}\left(\sum_{j \geq 0} (-1)^j S * S * \ldots * S \right).$$

One can check that this series converges to $\mathrm{Id} + \mathrm{Op}(Q)$ for some

$$Q \in \bigcap_{n \in \mathbb{N}} t^n \mathcal{C}^\infty(\mathbb{R}^+ \times M^2),$$

and hence

$$\beta^* H = \widetilde{H} + (\widetilde{H} * \beta^* Q).$$

In particular we see that

$$\beta^* H - \widetilde{H} \in \bigcap_{n \in \mathbb{N}} r^n \mathcal{C}^\infty(HM).$$

Theorem 40. *Let (M, g) be a compact Riemannian manifold, HM its heat space with blow-down map*

$$\beta : HM \longrightarrow \mathbb{R}^+ \times M^2$$

The pull-back of $H(t,x,y)$, the heat kernel of Δ_g, satisfies

$$\beta^* H \in r^{-m} \mathcal{C}^\infty(HM).$$

The trace of the heat kernel has an asymptotic expansion as $t \to 0$,

$$\mathrm{Tr}\, e^{-t\Delta} = \int_M H(t,x,x)\ \mathrm{dvol}_g \sim t^{-m/2} \sum_{k \geq 0} t^{k/2} a_k$$

with $a_0 = (4\pi)^{-m/2}\,\mathrm{Vol}(M)$ and each of the a_k are integrals over M of universal polynomials in the curvature of g and its covariant derivatives,

$$a_k = \int_M \mathcal{U}_k(x)\ \mathrm{dvol}_g.$$

The coefficients a_k are known as the **local heat invariants** of (M, g). One can use a symmetry argument to show that all of the a_k with k odd are identically zero, but the other a_k contain important geometric information.

Proof. The only thing left to prove is the statement about the coefficients a_k. Its clear from the construction above that the coefficients A_j are built up in a universal fashion from the expansion of the symbol of Δ. But as this symbol is precisely the Riemannian metric, and we know that its Taylor expansion at a point in normal coordinates has coefficients given by the curvature and its covariant derivatives, the result follows with

$$\mathcal{U}_k(x) = A_k(0, (1, 0), x).$$

□

6.3 Weyl's law

The following lemma is an example of a 'Tauberian theorem'.

Lemma 41 (Karamata)**.** *If μ is a positive measure on $[0, \infty)$, $\alpha \in [0, \infty)$, then*

$$\int_0^\infty e^{-t\lambda}\, d\mu(\lambda) \sim at^{-\alpha} \text{ as } t \to 0$$

implies

$$\int_0^\ell d\mu(\lambda) \sim \left[\frac{a}{\Gamma(\alpha+1)}\right] \ell^\alpha \text{ as } \ell \to \infty.$$

Proof. We start by pointing out that

$$t^\alpha \int e^{-t\lambda}\, d\mu(\lambda) = \int e^{-\lambda}\, d\mu_t(\lambda)$$

with $\mu_t(\lambda)(A) = t^\alpha \mu(t^{-1}A)$, so we can rewrite the hypothesis as

$$\lim_{t\to 0} \int e^{-\lambda}\, d\mu_t(\lambda) = a = \left[\frac{a}{\Gamma(\alpha)}\right] \int_0^\infty \lambda^{\alpha-1} e^{-\lambda}\, d\lambda$$

and the conclusion as

$$\lim_{t\to 0} \int \chi(\lambda)\, d\mu_t(\lambda) = \left[\frac{a}{\Gamma(\alpha)}\right] \int_0^\infty \lambda^{\alpha-1} \chi(\lambda)\, d\lambda$$

with $\chi(\lambda)$ equal to the indicator function for $[0, \ell]$. But this last equality is true for $\chi(\lambda) = e^{s\lambda}$ for any $s \in \mathbb{R}^+$, hence by density for any $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^+)$ and hence for Borel measurable functions, in particular for the indicator function of $[0, \ell]$. □

If we apply this to the measure $d\mu(\lambda) = \sum \delta(\lambda - \lambda_i)$, we can conclude.

Theorem 42 (Weyl's law)**.** *Let (M, g) be a compact Riemannian manifold and let $N(\Lambda)$ be the 'counting function of the spectrum' of the Laplacian,*

$$N(\Lambda) = \sum_{\substack{\lambda \in \mathrm{Spec}(\Delta) \\ \lambda < \Lambda}} \dim E_\lambda(\Delta_g).$$

As $\Lambda \to \infty$ we have

$$N(\Lambda) \sim \frac{\mathrm{Vol}(M)}{(4\pi)^{m/2}\Gamma(\frac{m}{2}+1)} \Lambda^{m/2}.$$

Hence the rate of growth of the eigenvalues of the Laplacian has geometric information: the volume of M and the dimension of M.

We also get geometric information from the local heat invariants. In fact it was pointed out by Melrose (for planar domains) and eventually shown in general by Gilkey, that they control the Sobolev norms of the curvature.

Theorem 43 (Gilkey[3])**.** *For $k \geq 3$,*

$$a_{2k} = (-1)^k \int_M A_k |\nabla^{k-2} R|^2 + B_k |\nabla^{k-2} \mathrm{scal}_g| \ \mathrm{dvol}_g + \int_M E_k \ \mathrm{dvol}_g$$

where A_k, B_k are positive constants and E_k is a universal polynomial in the curvature of g and its first $k-3$ covariant derivatives.

The explicit formulæ for the heat invariants quickly get out of hand. The first few are given by

$$(4\pi)^{m/2} a_0 = \mathrm{Vol}(M), \quad (4\pi)^{m/2} a_2 = \frac{1}{6} \int_M \mathrm{scal} \ \mathrm{dvol}_g$$

$$(4\pi)^{m/2} a_4 = \frac{1}{360} \int_M 5\mathrm{scal}^2 - 2|\mathrm{Ric}|^2 + 2|R|^2 \, \mathrm{dvol}_g$$

$$(4\pi)^{m/2} a_6 = \frac{1}{45360} \int_M (-142|\nabla \mathrm{scal}|^2 - 26|\nabla \mathrm{Ric}|^2 - 7|\nabla R|^2 + 35\mathrm{scal}^2 - 42\mathrm{scal}|\mathrm{Ric}|^2$$
$$+42\mathrm{scal}|R|^2 - 26 \mathrm{Ric}_{ij} \mathrm{Ric}_{jk} \mathrm{Ric}_{ki} - 20 \mathrm{Ric}_{ij} \mathrm{Ric}_{k\ell} R_{ikj\ell} - 8 \mathrm{Ric}_{ij} R_{ik\ell n} R_{jkln}$$
$$-24 R_{ijk\ell} R_{ijnp} R_{k\ell np}) \ \mathrm{dvol}_g$$

[3]For a proof see Gengqiang Zhou, *Compactness of isospectral compact manifolds with bounded curvatures.* Pacific J. Math. 181 (1997), no. 1, 187200.

There is a remarkable formula of Polterovich for the k^{th} heat invariant[4],

$$a_{2n}(x) = (4\pi)^{-m/2}(-1)^n \sum_{j=0}^{3n} \binom{3n+\frac{m}{2}}{j+\frac{m}{2}} \frac{1}{4^j j!(j+n)!} \Delta^{j+n} \left(d(x,y)^{2j}\right) \big|_{y=x}$$

however it is difficult to extract from here an explicit polynomial in the curvature.

6.4 Long-time behavior of the heat kernel

We know that the heat operator $e^{-t\Delta}$ is, for every $t > 0$, a bounded self-adjoint operator on $L^2(M)$. Hence its operator norm is equal to the modulus of its largest eigenvalue. In the same way

$$\|e^{-t\Delta} - \sum_{j=0}^{N} e^{-t\lambda_j}\mathcal{P}_{\lambda_j}\|_{L^2(M)\to L^2(M)} = e^{-t\lambda_{N+1}}.$$

Let us denote

$$e^{-t\Delta_N} = e^{-t\Delta} - \sum_{j=0}^{N} e^{-t\lambda_j}\mathcal{P}_{\lambda_j}$$

and point out that this is the heat operator of the Laplacian restricted to the orthogonal complement of span$\{\phi_0, \ldots, \phi_N\}$. In particular, the semigroup property applied to its integral kernel H_N yields

$$e^{-(s+t)\Delta_N} = e^{-s\Delta_N}e^{-t\Delta_N} \iff H_N(s+t,x,y) = \int_M H_N(s,x,z)H_N(t,z,y)\,dz.$$

Thus

$$H_N(t+2,x,y) = \int_{M^2} H_N(1,x,z_1)H_N(t,z_1,z_2)H_N(1,z_2,y)\,dz_1dz_2,$$

and we have

$$\begin{aligned} &|D_x^\alpha D_y^\beta H_N(t+2,x,y)| \\ &\quad \leq \|D_x^\alpha H_N(1,x,z_1)\|_{L^2(M)}\|e^{-t\Delta_N}\|_{L^2\to L^2}\|D_y^\beta H_N(1,z_2,y)\|_{L^2(M)} \leq C_{\alpha,\beta}e^{-t\lambda_{N+1}}. \end{aligned}$$

[4] See Gregor Weingart, *A characterization of the heat kernel coefficients.* `arXiv.org/abs/math/0105144`

Proposition 44. *Let (M, g) be a compact Riemannian manifold and $\{\phi_j\}$ an orthonormal eigenbasis of the Laplacian of g such that $\lambda_j \leq \lambda_{j+1}$. The series*

$$\sum_{j=0}^{N} e^{-t\lambda_j}\phi(x)\overline{\phi(y)}$$

converges uniformly to the heat kernel $H(t, x, y)$ in $\mathcal{C}^\infty(M)$. In particular, for any $t > 0$

$$\operatorname{Tr}(e^{-t\Delta}) = \sum e^{-t\lambda_j}$$

and for large t,

$$|\operatorname{Tr}(e^{-t\Delta} - \mathcal{P}_{\ker\Delta})| \leq e^{-t\lambda_1}.$$

Remark 7. One can show much more generally that the trace we defined on smoothing operators coincides with the functional analytic trace given by summing the eigenvalues.

6.5 Determinant of the Laplacian

The asymptotic expansion of the trace of the heat kernel allows us to define the determinant of the Laplacian. This is an important invariant with applications in physics, topology, and various parts of geometric analysis.

First we imitate the classical Riemann zeta function and define

$$\zeta(s) = \sum_{\substack{\lambda\in\operatorname{Spec}(\Delta)\\ \lambda\neq 0}} \lambda^{-s}$$

where eigenvalues are repeated in the sum according to their multiplicity. Here s is a complex variable, and Weyl's law shows that this sum converges absolutely (and hence defines a holomorphic function) if $\operatorname{Re}(s) > \frac{1}{2}\dim M$.

Notice that on the circle the eigenfunctions are $e^{i\theta n}$ with eigenvalue n^2, so the non-zero eigenvalues are the squares of the integers, each with multiplicity two. It follows that the ζ function of the circle is

$$\zeta(s) = 2\sum_{n\in\mathbb{N}} n^{-2s} = 2\zeta_{\text{Riemann}}(2s)$$

essentially equal to the Riemann zeta function!

Returning to general manifolds, we can use the identity

$$\mu^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty t^s e^{-t\mu}\,\frac{dt}{t}$$

valid whenever $\operatorname{Re}\mu > 0$ to write

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^s \ \operatorname{Tr}(e^{-t\Delta} - \mathcal{P}_{\ker\Delta}) \ \frac{dt}{t}$$

whenever $\operatorname{Re}(s) > \frac{1}{2}\dim M$. Indeed, Proposition 44 shows that the integral

$$\int_1^\infty t^s \ \operatorname{Tr}(e^{-t\Delta} - \mathcal{P}_{\ker\Delta}) \ \frac{dt}{t}$$

converges as $t \to \infty$, for any $s \in \mathbb{C}$, and Theorem 40 shows that the integral

$$\int_0^1 t^s \ \operatorname{Tr}(e^{-t\Delta} - \mathcal{P}_{\ker\Delta}) \ \frac{dt}{t}$$

converges as $t \to 0$, if $\operatorname{Re} s > m/2$. Moreover, it is straightforward to show that $\zeta(s)$ is a holomorphic function of s on the half-plane $\operatorname{Re} s > m/2$ where the integral converges.

Another use of Theorem 40 shows that the integral

$$\int_0^1 t^s \ \left(\operatorname{Tr}(e^{-t\Delta}) - t^{-m/2} \sum_{k=0}^N t^{k/2} a_k \right) \ \frac{dt}{t}$$

converges, and defines a holomorphic function, on the half plane $\operatorname{Re} s > \frac{m-N-1}{2}$. To incorporate the projection onto the null space, it suffices to replace a_k with

$$\widetilde{a}_k = \begin{cases} a_k + \dim\ker\Delta & \text{if } k = m \\ a_k & \text{otherwise} \end{cases}$$

On the other hand, for $\operatorname{Re} s > m/2$, we can explicitly integrate

$$\int_0^1 t^s \ t^{-m/2} \sum_{k=0}^N t^{k/2} \widetilde{a}_k \ \frac{dt}{t} = \sum_{k=0}^N \frac{\widetilde{a}_k}{s + (k-m)/2}$$

Thus, for any $N \in \mathbb{N}$, we have a function

$$\begin{aligned} \zeta_N(s) = \frac{1}{\Gamma(s)} \Bigg(& \int_1^\infty \operatorname{Tr}(e^{-t\Delta} - \mathcal{P}_{\ker\Delta}) \ \frac{dt}{t} \\ & + \int_0^1 t^s \ \left(\operatorname{Tr}(e^{-t\Delta} - \mathcal{P}_{\ker\Delta}) - t^{-m/2} \sum_{k=0}^N t^{k/2} \widetilde{a}_k \right) \ \frac{dt}{t} + \sum_{k=0}^N \frac{\widetilde{a}_k}{s + (k-m)/2} \Bigg) \end{aligned}$$

which is meromorphic on $\operatorname{Re} s > \frac{m-N-1}{2}$ and coincides with $\zeta(s)$ for $\operatorname{Re} s > m/2$. Since N was arbitrary we can conclude:

Theorem 45. *Let (M, g) be a compact Riemannian manifold of dimension m. The ζ function admits a meromorphic continuation to $\mathbb{C}$, with at worst simple poles located at $(m-k)/2$, $k \in \mathbb{N}$ and with residue*

$$\frac{\widetilde{a}_k}{\Gamma(\frac{m-k}{2})}$$

where $\widetilde{a}_k$ is the coefficient of $t^{(k-m)/2}$ in the short-time asymptotics of $\mathrm{Tr}(e^{-t\Delta} - \mathcal{P}_{\ker \Delta})$.

It follows easily that the meromorphic continuation of $\zeta(s)$ is always regular at $s = 0$. Notice that, if the Laplacian had only finitely many positive eigenvalues $\{\mu_1, \ldots, \mu_\ell\}$ then

$$\partial_s \left[\sum \mu_j^{-s}\right]_{s=0} = -\sum \log \mu_j = -\log \prod \mu_j \text{``} = -\log\det\Delta\text{''}.$$

Inspired by this computation, Ray and Singer defined

$$\det \Delta = e^{-\partial_s \zeta(s)\big|_{s=0}}$$

and, as mentioned above, this has found applications in topology, physics, and geometric analysis.

6.6 The Atiyah-Singer index theorem

In this section we will sketch how one can use the constructions of this lecture to prove the Atiyah-Singer index theorem. Recall that the objective is to find a topological formula for $\operatorname{ind} P$ where $P \in \Psi^k(M; E, F)$ is an elliptic operator acting between sections of Hermitian vector bundles E and F. A deep result known as Bott periodicity reduces the general problem to finding a formula for the index of a generalized Dirac operator as treated in Lecture 3, in fact it would be enough to handle twisted signature operators.

6.6.1 Chern-Weil theory

The connection with topology is through the theory of characteristic classes, which takes a vector bundle with some structure (e.g., an orientation or a complex structure) and defines cohomology classes.

Given a rank n $\mathbb{R}$-vector bundle $E \longrightarrow M$ and a connection ∇^E, recall (from section 2.5) that the curvature of ∇^E is a two-form valued in $\mathrm{End}(E)$,

$$R^E \in \Omega^2(M; \mathrm{End}(E)).$$

In a local trivialization of E, say by a local frame $\{s_j\}$, the connection is described by a matrix of one-forms

$$\nabla^E s_a = \omega_a^b s_b,$$

or in terms of the Christoffel symbols, $\omega_a^b = \Gamma_{ia}^b dx^i$. In this frame we have

$$(R^E)_a^b = d\omega_a^b - \omega_c^b \wedge \omega_a^c,$$

which we can express in matrix notation as $R^E = d\omega - \omega \wedge \omega$. Taking the exterior derivative of both sides, we find the 'Bianchi identity'

$$dR^E = -d\omega \wedge \omega + \omega \wedge d\omega = -R^E \wedge \omega + \omega \wedge R^E = [\omega, R^E]$$

Now let $P : \mathcal{M}_n(\mathbb{R}) \longrightarrow \mathbb{R}$ be an invariant polynomial, i.e., a polynomial in the entries of the matrix such that

$$P(A) = P(T^{-1}AT) \text{ for all } T \in \mathrm{GL}_n(\mathbb{R}).$$

Because of this invariance, it makes sense to evaluate P on sections of $\mathrm{End}(E)$ and in particular we get

$$P(R^E) \in \Omega^{\mathrm{even}}(M).$$

Since differential forms of degree greater than $\dim M$ are automatically zero, it makes equally good sense to evaluate any invariant power series on R^E.

Lemma 46. *If P is an invariant power series and R^E is the curvature of the connection ∇^E, then $P(R^E)$ is a closed differential form and its cohomology class is independent of the connection.*

Proof. By linearity, it suffices to prove this for P homogeneous, say of order q. For any q matrices, $A_1, \dots, A_q$, define $p(A_1, \dots, A_q)$ to be the coefficient of $t_1 t_2 \cdots t_q$ in $P(t_1 A_1 + \ldots + t_q A_q)$. Thus $p(A_1, \dots, A_q)$ is a symmetric multilinear function satisfying $p(A, \dots, A) = q! P(A)$ and moreover the invariance of P implies that

$$p(T^{-1}A_1 T, \dots, T^{-1}A_q T) = p(A_1, \dots, A_q) \text{ for all } T \in \mathrm{GL}_n(\mathbb{R}).$$

For any one-parameter family of matrices $T(s)$ in $\mathrm{GL}_n(\mathbb{R})$, with $T(0) = \mathrm{Id}$, we have

$$0 = \partial_s\big|_{s=0} p(T^{-1}A_1 T, \dots, T^{-1}A_q T) = \sum_{i=1}^{q} p(A_1, \dots, [A_i, T'(0)], \dots, A_q) = 0$$

and note that $T'(0)$ is an arbitrary element of $\mathcal{M}_n(\mathbb{R})$. We can apply this identity to see that $P(R^E)$ is closed,

$$\begin{aligned} dP(R^E) &= q!dp(R^E, \dots, R^E) = q(q!)p(dR^E, R^E, \dots, R^E) \\ &= q(q!)p([\omega, R^E], R^E, \dots, R^E) = 0. \end{aligned}$$

Next assume that $\widetilde{\nabla}^E$ is another connection on E. We know that their difference is an $\mathrm{End}(E)$ valued one-form,

$$\widetilde{\nabla}^E - \nabla^E = \alpha \in \Omega^1(M; \mathrm{End}(E)).$$

Define $\nabla^E_t = \nabla^E + t\alpha$, and note that this is a connection on E with curvature given locally by

$$R^E_t = d(\omega + t\alpha) - (\omega + t\alpha) \wedge (\omega + t\alpha) = R^E + t(d\alpha - [\omega, \alpha]) - \tfrac{1}{2}t^2[\alpha, \alpha].$$

A computation similar to that above yields

$$\partial_t P(R^E_t) = q(q!)dp(\alpha, R^E_t, \dots, R^E_t)$$

and so

$$P(R^E_1) - P(R^E_0) = q(q!) \int_0^1 dp(\alpha, R^E_t, \dots, R^E_t)\, dt = d\left(q(q!) \int_0^1 p(\alpha, R^E_t, \dots, R^E_t)\, dt \right).$$

(Another way of thinking about this is that the connections ∇^E_t can be amalgamated into a connection $\overline{\nabla}$ on the pull-back of E to $[0,1] \times M$, then the fact that $P(\overline{\nabla})$ is closed implies that $\partial_t P(R^E_t)$ is exact.) □

We will denote by $[P(E)]$ the cohomology class of $P(R^E)$. Recall that the symmetric functions on matrices of size n are functions of the elementary symmetric polynomials τ_j defined by

$$\det(t\,\mathrm{Id} + A) = t^n + \tau_1(A)t^{n-1} + \ldots + \tau_{n-1}(A)t + \tau_n(A),$$

so, e.g., $\tau_1(A) = \mathrm{Tr}(A)$, and $\tau_n(A) = \mathrm{Det}(A)$. Hence the cohomology class $[P(E)]$ is a a polynomial in the cohomology classes

$$[\tau_i(E)] \in H^{2i}(M).$$

Moreover, notice that if we choose a metric on E and a compatible connection ∇^E, then in a local orthonormal frame $\{s_a\}$ for E the matrix $(R^E)_a^b$ will be skew-symmetric and hence $\tau_i(R^E) = 0$ if i is odd. The non-vanishing classes

$$[\tau_{2i}(E)] \in H^{4i}(M)$$

are called the Pontryagin classes of E and they generate all of the characteristic classes of E.

If we restrict the class of matrices we consider then there are sometimes more examples of invariant polynomials. For instance, if we assume that E is an oriented bundle then we can assume that R^E is valued in skew-symmetric matrices and, if P is a polynomial on skew-symmetric matrices, we can define $P(R^E)$ as long as

$$P(T^{-1}AT) = P(A) \text{ for all } T \in \mathrm{SO}_n(\mathbb{R}).$$

This yields one new invariant, the Pfaffian characterized (up to sign) by the equation

$$\mathrm{Pff}(A)^2 = \mathrm{Det}(A).$$

Given a skew-symmetric matrix $A = (a_{ij})$, let

$$\alpha = \sum_{i<j} a_{ij} dx^i \wedge dx^j \in \Omega^2(\mathbb{R}^n)$$

then the Pfaffian of A is given by the n-fold wedge of α with itself,

$$\frac{1}{n!}(\alpha \wedge \alpha \wedge \cdots \wedge \alpha) = \mathrm{Pff}(A) dx^1 \wedge \cdots \wedge dx^n.$$

The Pfaffian of the oriented bundle E determines a cohomology class of degree $r = \mathrm{rank}(E)$,

$$[\mathrm{Pff}(E)] \in H^r(M),$$

which is reversed under a change of orientation, and vanishes if the rank of E is odd. It satisfies

$$[\mathrm{Pff}(E)] \wedge [\mathrm{Pff}(E)] = [\tau_r(E)] \in H^{2r}(M).$$

Now assume that the bundle E is a Hermitian vector bundle of complex rank n and for each Hermitian connection ∇^E define $c_i(R^E)$ by the equation

$$\det(t + iR^E) = t^n + c_1(R^E)t^{n-1} + \ldots + c_n(R^E).$$

The class of $[c_i(E)] \in H^{2i}(M)$ is known as the i^{th} Chern class of E. If F is a real vector bundle and $E = F \otimes \mathbb{C}$, then

$$[\tau_{2i}(F)] = [c_{2i}(E)] \in H^{4i}(M)$$

and, in this case, $[c_{2i+1}(E)] = 0$, but not in general.

We can specify an invariant polynomial by specifying its value on a diagonal or block-diagonal matrix. Thus for instance, if A is skew-adjoint, it can be diagonalized over $\mathbb{C}$ with eigenvalues μ_j. We define the Chern polynomial $\mathrm{Ch}(A) = \sum e^{\mu_j}$ and the Todd polynomial $\mathrm{Td}(A) = \prod \frac{\mu_j}{(1-e^{-\mu_j})}$ and then define, for any complex Hermitian vector bundle E,

$$\text{Chern character of } E = \mathrm{Ch}(R^E), \quad \text{Todd genus of } E = \mathrm{Td}(R^E).$$

On the other hand, if B is skew-symmetric real valued matrix then, over $\mathbb{R}$, we can write it in block-diagonal form where each block is either of the form

$$\begin{pmatrix} 0 & -\mu_j \\ \mu_j & 0 \end{pmatrix}$$

or is the 1-by-1 zero matrix. For these matrices we define the Hirzebruch L-polynomial by $L(B) = \prod \frac{\mu_j}{\tanh(\mu_j)}$ and the A-roof genus by $\widehat{A}(B) = \prod \frac{\mu_j}{2\sinh(\frac{\mu_j}{2})}$ (note that, as these are even functions, the sign ambiguity in the definition of μ_j is unimportant) and then, for any real vector bundle,

$$L\text{-polynomial of } E = L(R^E), \quad A\text{-roof genus of } E = \widehat{A}(R^E).$$

6.6.2 The index theorem

Let $E \longrightarrow M$ be a $\mathbb{Z}_2$-graded Dirac bundle, let γ be its involution, and $\eth_E \in \mathrm{Diff}^1(M;E)$ the associated generalized Dirac operator. The construction of the heat kernel above extends easily to Laplace-type operators such as $\eth_E^2$,

$$e^{-t\eth_E^2} = \begin{pmatrix} e^{-t\eth_E^-\eth_E^+} & 0 \\ 0 & e^{-t\eth_E^+\eth^-} \end{pmatrix}$$

and establishes short-time asymptotic expansions

$$\mathrm{Tr}(e^{-t\eth_E^-\eth_E^+}) \sim t^{-m/2}\sum a_{k,E}^+ t^{k/2}, \quad \mathrm{Tr}(e^{-t\eth_E^+\eth_E^-}) \sim t^{-m/2}\sum a_{k,E}^- t^{k/2}$$

with each coefficient given by the integral of a polynomial in the curvature of g, the curvature of ∇^E, and the covariant derivatives of these curvatures,

$$a_{k,E}^\pm = \int_M \mathcal{U}_{k,E}^\pm(x)\ \mathrm{dvol}_g\,.$$

It is easy to see (exercise 5) that the non-zero spectrum of the composition of two operators AB, coincides with the non-zero spectrum of BA, and that the corresponding eigenspaces have the same dimension. One remarkable consequence pointed out by McKean and Singer is that

$$\mathrm{Tr}(e^{-t\eth_E^-\eth_E^+}) - \mathrm{Tr}(e^{-t\eth_E^+\eth_E^-}) = \mathrm{ind}(\eth_E^+) \text{ for all } t > 0.$$

(One can also prove this by showing that the derivative of the left hand side with respect to t vanishes, and then taking the limit of the left hand side as $t \to \infty$.) They noted that this gives a very non-explicit formula for the index,

$$\mathrm{ind}(\eth_E^+) = \int_M \mathcal{U}_{m,E}^+(x) - \mathcal{U}_{m,E}^-(x) \ \mathrm{dvol}_g, \tag{6.1}$$

and conjectured that after a 'fantastic cancellation', the universal polynomial

$$\mathcal{U}_{m,E}^+(x) - \mathcal{U}_{m,E}^-(x)$$

depends only on the curvatures and on no covariant derivatives.

Note that the formula (6.1) already gives interesting information about the index of a Dirac-type operator. For instance, it vanishes in odd dimensions, it is additive for connected sums, and multiplicative for finite covers.

The fantastic cancellation, now known as the *local index theorem* was subsequently shown in a couple of different ways. First Patodi was able to show that the cancellation takes place for a couple of specific operators by a direct computation. Then Gilkey gave a proof, simplified by Atiyah-Bott-Patodi, that first characterizes the form-valued invariants of a Riemannian structure and Hermitian bundle that are 'natural' in that they are invariant under pull-back, as precisely the elements of the ring generated by the Pontrjagin forms of TM and the Chern forms of E. The invariance properties of $\mathcal{U}_{m,E}^+(x) - \mathcal{U}_{m,E}^-(x)$ are enough to show that it is natural, and then evaluation of some simple examples determines the polynomial explicitly.

The other approach allows a direct computation $\mathcal{U}_{m,E}^+(x) - \mathcal{U}_{m,E}^-(x)$ using the 'supersymmetry' of Clifford algebras. This was discovered by Getzler who showed that after taking the supersymmetry into account, one could use the classical formula of Mehler for the heat kernel of the harmonic oscillator. In one dimension this says that

$$P_1 = -\partial_x^2 + x^2 \implies$$
$$e^{-tP_1}(x,y) = \frac{1}{\sqrt{2\pi \sinh 2t}} \exp\left(-\frac{1}{2\sinh 2t}(x^2\cosh 2t - 2xy + y^2\cosh 2t) \right),$$

but this can be generalized to higher dimensional Laplacians acting on vector bundles.

Applied to the Dirac operator on a spin manifold, this yields

$$\mathcal{U}^+_{m,\not{S}}(x) - \mathcal{U}^-_{m,\not{S}}(x) = \mathrm{Ev}_m \left(\det{}^{1/2} \left(\frac{R/4\pi i}{\sinh(R/4\pi i)} \right) \right)$$

and hence

$$\mathrm{ind}(\eth^+) = \int_M \widehat{A}(TM).$$

For a generalized Dirac operator associated to a $\mathbb{Z}_2$-graded Dirac bundle over an even dimensional manifold, recall that we have

$$\mathrm{End}(E) = \mathbb{C}l(T^*M) \otimes \mathrm{End}_{\mathbb{C}l(T^*M)}(E), \quad R^E = \tfrac{1}{4} cl\left(\widetilde{R}\right) + \widehat{R}^E$$

with $\widehat{R}^E$, the twisting curvature, a section of $\mathrm{End}_{\mathbb{C}l(T^*M)}(E)$. Similarly the grading γ of E decomposes into $cl\,(\Gamma) \otimes \widehat{\gamma}$ with $\widehat{\gamma} \in \mathrm{End}_{\mathbb{C}l(T^*M)}(E)$. The local index theorem in this case is the fact that

$$\mathcal{U}^+_{m,E}(x) - \mathcal{U}^-_{m,E}(x) = \mathrm{Ev}_m \left(\det{}^{1/2} \left(\frac{\widetilde{R}/4\pi i}{\sinh(\widetilde{R}/4\pi i)} \right) \mathrm{tr}\left(\widehat{\gamma} \exp(i\widehat{R}^E/2\pi) \right) \right).$$

The differential form $\mathrm{tr}\left(\widehat{\gamma}\exp(i\widehat{R}^E/2\pi)\right)$ is known as the *relative Chern form of* E, it is closed and its cohomology class is independent of the choice of Clifford connection on E and will be denoted $\mathrm{Ch}'(E)$. The index theorem in this case is

$$\mathrm{ind}(\eth^+_E) = \int_M \widehat{A}(TM)\,\mathrm{Ch}'(E).$$

6.7 Examples

For the Gauss-Bonnet operator, recall that we have

$$\eth^{\mathrm{even}}_{\mathrm{dR}} : \Omega^{\mathrm{even}}(M) \longrightarrow \Omega^{\mathrm{odd}}(M)$$

and that the index of $\eth^{\mathrm{even}}_{\mathrm{dR}}$ is equal to the Euler characteristic of M. The index theorem in this case is the Chern-Gauss-Bonnet theorem

$$\chi(M) = \mathrm{ind}(\eth^{\mathrm{even}}_{\mathrm{dR}}) = \int_M \mathrm{Pff}(TM).$$

For the signature operator, recall that we use the Hodge star to split

$$\Lambda^*_{\mathbb{C}} T^*M = \Lambda^*_+ T^*M \oplus \Lambda^*_- T^*M$$

and then the restriction of the de Rham operator to $\Lambda^*_+ T^*M$ is the signature operator

$$\eth^+_{\text{sign}} : \Lambda^*_+ T^*M \longrightarrow \Lambda^*_- T^*M.$$

The index of $\eth^+_{\text{sign}}$ is the signature of the manifold M. The index theorem in this case is the Hirzebruch signature theorem

$$\operatorname{sign}(M) = \operatorname{ind}(\eth^+_{\text{sign}}) = \int_M L(M).$$

For the $\overline{\partial}$ operator on a Kähler manifold, recall that we have a splitting

$$\Lambda^k T^*M \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q} M$$

with respect to which d splits into

$$d = \partial + \overline{\partial} : \Omega^{p,q} M \longrightarrow \Omega^{p+1,q} M \oplus \Omega^{p,q+1}(M)$$

and then operator

$$\eth^+_{RR} = \sqrt{2}\,(\overline{\partial} + \overline{\partial}^*) : \Omega^{0,\text{even}} M \longrightarrow \Omega^{0,\text{odd}}(M)$$

is a generalized Dirac operator whose index is the Euler characteristic of the Dolbeault complex. The index theorem in this case is the Grothendieck-Riemann-Roch theorem

$$\operatorname{ind}(\eth^+_{RR}) = \int_M \operatorname{Td}(TM \otimes \mathbb{C}).$$

For a general elliptic pseudodifferential operator $P \in \Psi^*(M; E, F)$, choose a connection ∇^E with curvature $-2\pi i \omega_E$ and a connection ∇^F with curvature $-2\pi i \omega_F$. Define

$$\omega(t) = (1-t)\omega_E + t\sigma(P)^{-1}\omega_F \sigma(P) + \tfrac{1}{2\pi i} t(1-t)(\sigma^{-1}(P)\nabla\sigma(P))^2$$

and then set

$$\widetilde{\operatorname{Ch}}(\sigma(P)) = -\frac{1}{2\pi i}\int_0^1 \operatorname{Tr}(\sigma^{-1}(P)\nabla\sigma(P) e^{\omega(t)})\, dt \in \Omega^{\text{odd}}(\mathbb{S}^*M)$$

This is the 'odd Chern form' defined by the symbol. The Atiyah-Singer index theorem is the formula

$$\operatorname{ind}(P) = \int_{\mathbb{S}^*M} \widetilde{\operatorname{Ch}}(\sigma(P))\operatorname{Td}(TM \otimes \mathbb{C}).$$

6.8 Exercises

Exercise 1.
Show that zero is an eigenvalue of infinite multiplicity for $d : \Omega^*(M) \longrightarrow \Omega^*(M)$, that d has no other eigenvalue, and that there is an infinite dimensional space of forms that are not in this eigenspace.

Exercise 2.
Verify directly that the spectrum of the Laplacian on $\mathbb{R}$ is $[0, \infty)$ and that there are no eigenvalues.

Exercise 3.
Compute the spectrum of the Laplacian on the flat torus $\mathbb{R}^2/\mathbb{Z}^2$. Compute the spectrum of the Laplacian on 1-forms on the flat torus.

Exercise 4.
Compute the spectrum of the sphere $\mathbb{S}^2$ with the round metric.

Exercise 5.
Show that AB and BA have the same spectrum, including multiplicity, except possibly for zero.

Exercise 6.
Show that $E_\lambda(\Delta_k)$ form an acyclic complex.

Exercise 7.
Use the Fourier transform to show that the integral kernel of the heat operator on $\mathbb{R}^m$ is given by

$$\mathcal{K}_{e^{-t\Delta}}(t, \zeta, \zeta') = \frac{1}{(4\pi t)^{m/2}} e^{-\frac{|\zeta - \zeta'|^2}{4t}}.$$

Exercise 8.
Use the heat kernel to give an alternate proof that a de Rham class contains a harmonic form.

6.9 Bibliography

A great place to read about spectral theory is the book *Perturbation theory for linear operators* by Kato, I also like *Functional analysis* by Rudin.

Our construction of the heat kernel is due to Richard Melrose and can be found in Chapter 7 of *The Atiyah-Patodi-Singer index theorem.* One should also look at Chapter 8 to see how to carry out Getzler rescaling geometrically.