Math 514

Last Time: $\text{Diff}^k(M;E,F) \subseteq \Psi^k(M;E,F)$

$v \in \mathbb{R}$, $A \in \Psi^k(M;E,F)$ define a continuous map $\mathcal{H}^k(M,E) \rightarrow \mathcal{H}^k(M,E)$

If $\sigma_R(A)(\xi)$ is invertible for $\xi \neq 0$, then $A$ is elliptic.

If $\xi \in \mathcal{H}^{-r}(M;F,E)$ then $A - \text{Id} \in \Psi^{-r}(M;E)$

This Time: Prove the Hodge Theorem

Today: Let $(M,g)$ be a closed Riemannian manifold $E,F \rightarrow M$ with $\gamma$ Hermitian metrics, $r > 0$

**Claim (Elliptic estimate)** Let $A \in \Psi^k(M;E,F)$ be elliptic.

For each $m \in \mathbb{R}$ there exists $C > 0$ such that

If $v \in \mathcal{H}^k(M;E)$ for some $k$ and $A v \in \mathcal{H}^k(M;E)$

Then $v \in \mathcal{H}^r(M;E)$ and

$$\|v\|_{\mathcal{H}^r} \leq C(\|Av\|_{\mathcal{H}^k} + \|v\|_{\mathcal{H}^k})$$

**Proof**

Let $B \in \Psi^{-r}(M;E,F)$ be a parametrix for $A$. Then $B = B A - \text{Id} \in \Psi^{-r}(M;E,F)$

Then $v = B A v - B \xi \in \mathcal{H}^{r}(M;E)$

Also $B : \mathcal{H}^k(M;E,F) \rightarrow \mathcal{H}^{r}(M;E,F)$ is continuous, so

$$\|B A v\|_{\mathcal{H}^r} \leq C \|A v\|_{\mathcal{H}^k}$$

Also $R : \mathcal{H}^k(M;E,F) \rightarrow \mathcal{H}^{r}(M;E,F)$ is continuous, so

$$\|R v\|_{\mathcal{H}^r} \leq C \|v\|_{\mathcal{H}^k}$$
Remark: There's a "low tech" way of obtaining the elliptic estimate without introducing \( \Phi_0 \).

Consider something like \( \text{Id} + \Delta \in \text{Diff}^2(M) \).

At a given point \( \mathfrak{p} \in M \) we can choose coordinates so that

the metric, at least \( \mathfrak{p} \), is the Euclidean metric.

Locally, coefficients at this point we have \( \text{Id} + \Delta = \text{Id} + \Delta \mathfrak{p} \in \text{Diff}^2(M) \).

This now satisfies \( (\text{Id} + \Delta \mathfrak{p})(f) = \Phi^\mathfrak{p} \circ (1 + \mathfrak{p})^{-1} \Phi^\mathfrak{p}(f) \).

So the inverse is given by \( (\text{Id} + \Delta \mathfrak{p})^{-1}(f) = \Phi^\mathfrak{p} \circ (1 + \mathfrak{p})^{-1} \Phi^\mathfrak{p}(f) \).

This is obviously well-behaved as a map between Sobolev spaces.

For \( v \in H^s(M) \) we have

\[ \|v\|_{H^s} \leq C \|\text{Id} + \Delta \mathfrak{p}\|_{L^\infty} \|v\|_{H^s} \]

Back on \( M \), if we localize at a coordinate chart instead of \( \mathfrak{p} \), then

\[ (\text{Id} + \Delta \mathfrak{p})v = \left[ \text{Id} + \Delta \mathfrak{p}_2 + (\Delta \mathfrak{p} - \Delta \mathfrak{p}_2) \right]v \]

So \( v = (\text{Id} + \Delta \mathfrak{p}_2)^{-1}(\text{Id} + \Delta \mathfrak{p} + \Delta \mathfrak{p} - \Delta \mathfrak{p}_2)v \)

\[ \|v\|_{H^s(M)} \leq C \|\text{Id} + \Delta \mathfrak{p}\|_{L^\infty} \|v\|_{H^s(M)} + C' \|v\|_{H^s(M)} \]

The elliptic estimate on \( M \) is obtained by patching these together.

"Gluing together" the inverses of \( \text{Id} + \Delta \mathfrak{p}_2 \) produces a pseudodifferential metric for \( \text{Id} + \Delta \mathfrak{p}_2 \).

Thus this method is not too far from the pseudodifferential approach.
Corollary If $M$ is closed & $A \in \mathcal{L}(\mathcal{H}^\omega(M;E,F))$ elliptic, $r > 0$

1) $\text{Ker} (A : H^{\omega r}(M;E) \to H^\omega(M;E)) \leq C^\omega(M;E)$

is finite dimensional & independent of $r$.

2) For any $s > 0$, $A(H^{\omega r}(M;E))$ is a closed subspace of $H^s(M;E)$.

Proof

1) If $Au \in C^\omega(M;E) = \bigwedge H^\omega(M;E)$

then by the theorem we $\bigwedge H^{\omega r}(M;E) = C^\omega(M;E)$

The elliptic estimate, applied to elements of $\text{Ker} A$

gets that $\exists C > 0$, $\|u\|_{H^{\omega r}} \leq C \|u\|_{H^\omega}$.

Thus the identity map is continuous as a map

$I : (\text{Ker} A, \|\cdot\|_{H^{\omega r}}) \to (\text{Ker}, \|\cdot\|_{H^\omega})$.

Since $A$ is linear, $H^{\omega r}(M;E) \to H^\omega(M;E)$ is a continuous operator

any sequence in $\text{Ker} A$ yields a sequence in $H^{\omega r}(M;E)$

has a subsequence that converges in $H^\omega$ hence in $H^{\omega r}$.

Let $\{e_j\}$ be a basis of $\text{Ker} A$, orthonormal in $H^{\omega r}$.

If $\{e_j\}$ were infinite, it would have a convergent subsequence,

but $\|e_j - e_k\|^2 = (e_j, e_j) + (e_k, e_k) = 2$ so $\{e_j\}$ is finite.

2) Let $(u_j)$ be a sequence in $H^{\omega r}(M;E)$

such that $Au_j \rightarrow u$ in $H^\omega(M;E)$ w.r.t. $\text{Im} A$.

Assuming, as we may, that $u \perp \text{Ker} A$, we'll show that $u$

have a convergent subsequence.
First assume that \((u_j)\) has a subsequence that is bold in \(H^{5\frac{1}{2}}(M;E)\).

Then \((u_j)\) has a subseq that converge in \(H^5(M;E)\)

\(\Delta\) by the elliptic estimate \(\|u_j\|_{H^5} \leq C(\|u_j\|_{H^{5\frac{1}{2}}} + \|u_j\|_{E})\)

a subsequence that converge in \(H^5\)

say \(u_j \to u_0\). Continuity of \(A = \) \(A_{u0} = \nabla u_0\).

Otherwise \(\|u_j\|_{H^{5\frac{1}{2}}} \to \infty\)

Let \(w_j = u_j\) \(\in H^{5\frac{1}{2}}(M;E)\) \& \(A_{w_j} = A_{u_j} \to 0\) in \(H^5\)

\(\|w_j\|_{H^{5\frac{1}{2}}} = \infty\).

We can apply the proceeding argument since \(\|u_j\|_{H^{5\frac{1}{2}}} = 1\).

\(\Delta\) conclude that \((w_j)\) has a subsequence that converge in \(H^{5\frac{1}{2}}\)

say \(w_j \to w_0\).

\(\nabla w_0 = 0\) \& \(w_0 \in \text{ker } A\) \(\perp \Rightarrow w_0 = 0\)

but \(\|w_0\|_{H^{5\frac{1}{2}}} = 1\).

This \((u_j)\) must have a bold subseq.