Math 514

Last Time: Kodaira embedding theorem
This Time: Sobolev spaces

Although we're mostly interested in the Hodge decomposition for smooth differential forms, it is easier to establish the Hodge decomposition for $L^2$ differential forms and then deduce the smooth version from that.

Differential operators are not a priori defined on $L^2$, so the first thing we should do is describe Sobolev spaces where they are defined.

The key concept is the "weak derivative" of a function. If $f \in C^1(\mathbb{R}) = C(\mathbb{R}, \mathbb{R})$ then for any $\phi \in C_c^\infty(\mathbb{R})$ we have $\int \phi'(x) f'(x) \, dx = -\int \phi(x) f''(x) \, dx$ so we can identify $f'(x)$ with the functional

$$C^\infty_c(\mathbb{R}) \ni \phi \quad \mapsto \quad -\int \phi'(x) f''(x) \, dx$$

The advantage of using this functional expression is that it makes sense for $f'$ that are not differentiable.

We refer to this functional as the weak derivative of $f$.

If this functional extends to all $\phi \in L^1$ then by Riesz representation, $f \in L^1$ s.t. $\nabla (\phi) = \langle \psi, h \rangle = \int \phi(x) f(x) \, dx$.

That is, $f \in L^1$ s.t. $-\int \phi'(x) f(x) \, dx = \int \phi(x) f(x) \, dx$ if we identify $\nabla \psi = h$ and call $\psi$ the weak derivative of $f$. Dec 4, 2020
Consider the function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$

If $\psi \in C_c^\infty(\mathbb{R})$ then the weak derivative of $f$

applied to $\psi$ is

$$- \int_\mathbb{R} \psi'(x) f(x) \, dx = - \int_\mathbb{R} \psi'(x) |x| \, dx - \int_\mathbb{R} \psi(x) \, dx$$

$$= - \int_\mathbb{R} \psi(x) \, dx + \int_\mathbb{R} \psi(x) \, dx = \int_\mathbb{R} \psi(x) \, dx$$

Thus the weak derivative of $|x|$ is $\text{sign}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$

We can keep going, the weak derivative of $\text{sign}(x)$

is the functional $\text{sign}: C_c^\infty(\mathbb{R}) \to \mathbb{R}$

$$\text{sign}(\psi) = \int_\mathbb{R} \psi(x) \, dx$$

$$\Lambda(\psi) = \int_\mathbb{R} \psi(x) \, dx + \int_\mathbb{R} \psi(x) \, dx = 2 \int_\mathbb{R} \psi(x) \, dx = \int_\mathbb{R} \psi(x) \, dx$$

We say that the weak derivative of $\text{sign}(x)$ is

Hence the delta "function" at $0$. Of course it isn't

a function, it's a functional.

The functionals on $C_c^\infty(\mathbb{R})$ are known as distributions.

Similarly, if we let $K(\mathbb{R}^n)$ be a multi-index

we say that we let $K(\mathbb{R}^n)$ is $D^\alpha$ computed weakly (distributionally)

if, for all $\psi \in C_c^\infty(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} D^\alpha \psi \, dx = (-1)^|\alpha| \int_{\mathbb{R}^n} \psi \, dx$$

(In general we define the weak derivative as a functional.)

The $k^{th}$ Sobolev space, for $k \in \mathbb{N}$,

$$H^k(\mathbb{R}^n) = \{ f \in L^2(\mathbb{R}^n) : \forall \alpha \leq k, D^\alpha f \in L^2(\mathbb{R}^n) \}$$
Using the Fourier transform
$$\hat{f}(\xi) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x} \, dx$$

an equivalent definition is
$$H^k(\mathbb{R}^n) = \{ f \in L^2(\mathbb{R}^n) : p(\xi) \hat{f}(\xi) \in L^2(\mathbb{R}^n) \text{ for polynomial } p \text{ of degree } \leq k \}$$

Sobolev spaces are Hilbert spaces with respect to
$$f, \tilde{f} \in H^k(\mathbb{R}^n), \quad (f, \tilde{f})_{H^k} = \sum_{|\alpha| \leq k} \int \partial^\alpha f \overline{\partial^\alpha \tilde{f}} \, dx$$

An equivalent Hilbert space structure is given by
$$\langle f, \tilde{f} \rangle_{H^k} = \int_{\mathbb{R}^n} \frac{\partial f}{\partial x} \overline{\frac{\partial \tilde{f}}{\partial x}} \left( 1 + |x|^2 \right)^{k/2} \, dx$$

This inner product makes sense for $k \in \mathbb{R}$

& the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the resulting norm is $H^k(\mathbb{R}^n)$, $k \in \mathbb{R}$

If $M$ is a compact manifold $\mathcal{E} \to M \times \mathbb{R}^n$

we choose a Riemannian metric on $M$ & a Hermitian metric on $\mathcal{E}$

& use these to define an $L^2$-inner product on sections of $\mathcal{E}$

The topological space $L^2(M; \mathcal{E})$ is independent of the choice.

We can define $H^k(M; \mathcal{E})$ in two equivalent ways:

1. Pick a finite cover of charts trivializing $\mathcal{E}$

2. subordinate partition of unity $\xi_j$ & declare $u \in H^k(M; \mathcal{E})$ if $u_j, u \in H^k(\mathbb{R}^n; \mathcal{E})$ \& $\| u \|_{H^k}^2 = \sum_j \| u_j \|_{H^k}^2$
(ii) Pick a metric connection \( \nabla \). Take
\[
\|u\|_{k;E} = \frac{1}{\sqrt{n!}} \int \sum_{|\alpha| \leq k} u_{\alpha} \cdot \Delta^\alpha u^2 \, dx
\]
(if \( k \in \mathbb{N} \)).

Clearly \( H^k(M;\mathbb{E}) \subseteq H^{k'}(M;\mathbb{E}) \) if \( k \geq k' \).

If \( M \) is compact then the inclusion \( H^k(M;\mathbb{E}) \subseteq H^{k'}(M;\mathbb{E}) \)

is a compact operator whenever \( k \geq k' \) (Rellich's Theorem)

(Thus is not an L-version of Arzela-Ascoli).

Recall Fourier inversion says that if
\[
\mathcal{F}(\omega)(x) = \frac{1}{(2\pi)^n} \int \mathcal{F}(\xi) e^{i\xi \cdot x} \, d\xi
\]

then \( \mathcal{F}^* \) is the inverse of \( \mathcal{F} \) (and its adjoint)

as maps between \( \mathcal{L}^1(\mathbb{R}^n) \to \mathcal{L}^\infty(\mathbb{R}^n) \)

or \( \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n) \).

Here \( \mathcal{S}(\mathbb{R}^n) = \{ f \in C^\infty(\mathbb{R}^n) : \forall \alpha \exists \beta \in \mathbb{N}^n, \forall M, M' \in \mathbb{N} \text{ such that } \| \partial_x^\alpha D_x^\beta f \|_{L^\infty} < M \} \} \)

Let \( \mathcal{S}'(\mathbb{R}^n) \) be the dual space of \( \mathcal{S}(\mathbb{R}^n) \).

These are known as the tempered distributions.

For any two functions \( f, h \in \mathcal{S}(\mathbb{R}^n) \)

the \( L^2 \)-pairing satisfies \( (\mathcal{F}(f),h) = (f, \mathcal{F}^*(h)) \).

So if \( \Lambda \) is a tempered distribution

we define \( \mathcal{F}(\Lambda) \) to be the tempered distribution

\[
\mathcal{F}(\Lambda)(f) = \Lambda(\mathcal{F}(f))
\]

Thus if \( \Lambda \) is given by \( \Lambda(h) = (h,f) \)

then \( \mathcal{F}(\Lambda)(h) = \Lambda(\mathcal{F}^*h) = (\mathcal{F}^*h, f) = (h, \mathcal{F}^*(f)) \).

Similarly extending \( \mathcal{F}^* \) to tempered distributions we see

\( \mathcal{F}^* \) holds on \( \mathcal{S}' \).
If \( s > \frac{n}{2} + k \), then \( H^s(\mathbb{R}^n) \subseteq C^k(\mathbb{R}^n) \land L^\infty(\mathbb{R}^n) \)

\[
|f(x)| \leq \frac{1}{(2\pi)^{\frac{n}{2}}} \int \frac{\partial^s f(\xi)}{\partial^s \xi^s} e^{ix \cdot \xi} d\xi \\
\leq \frac{1}{(2\pi)^{\frac{n}{2}}} \int |\partial^s f(\xi)| d\xi \\
\leq \frac{1}{(2\pi)^{\frac{n}{2}}} \int \|\partial^{k+j} f(\xi)\|_1 (1 + |\xi|)^{s-k-j} d\xi \\
\underbrace{\int (1 + |\xi|)^{-s} d\xi}_{\text{finite precisely when } s > \frac{n}{2}}
\]

Similarly, if \( s > \frac{n}{2} + k \), \( k \in \mathbb{N} \) then \( H^s(\mathbb{R}^n) \subseteq C^k(\mathbb{R}^n) \)

In particular, \( H^s(\mathbb{R}^n) \subseteq C^0(\mathbb{R}^n) \)