Last Time: Positive Differential Form

A real diff form of type $\omega(1,1)$ is positive

if the bilinear form $(\omega, \omega) : \Omega^1(M) \times \Omega^1(M) \to \mathbb{R}$ is positive

If $\omega$ is closed $\omega$ is positive

then it is the form of a K-Herm structure on $M$

$c \in H^0(M; \mathbb{C})$ is positive if it has a positive rep

of a holomorphic bundle $E \to M$ is positive if $c_2(E)$ is positive

This Time: Kähler vanishing theorem

Lemma: If $M$ is a closed K-Herm mfd $\omega \in \Omega^1(M; \mathbb{C})$

has a positive rep then it does not have a negative rep.

If on a closed mfd a closed positive diff form $\omega$ is not

exact (since the volume of the connected K-Herm metric is $\int_M \omega^n$)

However if $\omega$ were to have a pos rep $\omega_+$ & a neg rep $\omega_-$

then $\omega_+ - \omega_-$ would be both positive & exact.

Lemma: If $M$ is K-Herm $\omega \in \Omega^1(M)$ is a holomorphic bundle

then $\omega > 0$ iff there is a Herm metric on $E$, $h \in \mathfrak{h}$ where

Chern connection curvature $R^\omega$ satisfies $iR^\omega > 0$

Since $c_2(E) = \left[ c_2 \left( \frac{i}{2\pi} R^\omega \right) \right]$

it's clear that $iR^\omega > 0 \Rightarrow E > 0$
Conversely, suppose $E > 0 \mathrm{ i.e. } c_1(E) > 0$

We can endow $M \times E$ a K-Herm structure $(M, g, J, \omega)$

$s.t. \ [\omega] \in 2\pi c_1(E)$

Let $\tilde{\omega}$ be any Hermitian metric on $E$

$\nabla^{\tilde{\omega}}$ the curvature of its Chern connection

Recall from last time that $\tilde{\omega} = \tilde{\omega} \log \tilde{\omega}$ ($\tilde{\omega} = \tilde{\omega}$)

where $\psi$ is any non-zero local section of $E$

In particular \ $\tilde{\omega} = \tilde{\omega}$ (and recall $\tilde{\omega} = \tilde{\omega}$)

Now $[i \tilde{\omega}] = 2\pi c_1(E) = [\omega]$

so by the \(\tilde{\omega}\)-lemma, $\omega - i \tilde{\omega} = i \tilde{\omega} \psi$ for some $\psi$

Let $h = \tilde{\omega} \psi$

The curvature of the Chern connection of $h$ satisfies

$i \tilde{\omega} \psi = i \tilde{\omega} \log h = i \tilde{\omega} \psi + (\omega - i \tilde{\omega} \psi) = \omega > 0$

Proof: Let $M$ be K-Herm, $E = M$ positive scalar line bundle

endowed by $h \in \text{End } E$ for some $h \in \mathbb{R}$

Then if $(M, g, J, \omega = h \tilde{\omega})$ is the resulting K-Herm structure

we have $\mathcal{D}_{\tilde{\omega}} - \Delta_{\tilde{\omega}} = \frac{1}{h} (\mu + n - 1)$ ($n = \text{dim } M$)

We've already seen that $\Delta_{\tilde{\omega}} - \Delta_{\tilde{\omega}} = [i \tilde{\omega} \text{End } E, \Lambda]$

and in this setting this equals $\frac{1}{h} [L, \Lambda] = \frac{1}{h} (N - n)$
Then (Kodaira vanishing theorem)

Let $M$ be a closed $K$-Hausdorff manifold and $E \to M$ a holomorphic bundle.

1) if $E > 0$ then $H^{n,q}(M;E) = 0$ if $p + q > n$

2) if $E < 0$ then $H^{p,q}(M;E) = 0$ if $p + q < n$

Proof:

Some duality shows that (i) implies (ii) since $E > 0 \implies E^\ast < 0$

Let's prove (ii):

Pick a Hermitian metric on $E$ s.t. the curvature of its

Chern connection satisfies $iR^g < 0$

Pick the corresponding $K$-Hausdorff structure on $M$, $(H, J, \omega = -i\Omega^g)$

If $\alpha$ is a $\Delta_{\bar{\partial}_E}$-harmonic form $\gamma \in \Omega^N$ of type $(p,q)$

then $(\Delta_{\bar{\partial}_E} - \Delta_{\bar{\partial}_E}) \alpha = -\Delta_{\bar{\partial}_E} \alpha = -(p + q - n) \alpha$

Thus $(n - p - q)(\alpha, \alpha) = -(\Delta_{\bar{\partial}_E} \alpha, \alpha)$

$= -(\bar{\partial}_E^* \partial_E \alpha, \alpha)$

$= - \left[ \partial_E^* \partial_E \alpha + \bar{\partial}_E^* \bar{\partial}_E \alpha \right]$

$\leq 0$

Since $(\alpha, \alpha) = \| \alpha \|^2 > 0$, either $(n - p - q) \leq 0$ or $\alpha = 0$

So since $H^{p,q}(M;E) \cong Ker \Delta_{\bar{\partial}_E} |_{\Omega^N(M;E)}$, (ii) follows.

Note: Let a diff form of type $(p,q)$ $\gamma \in \Omega^N$ be

In the case of a diff form of type $(0,q)$ $\gamma \in \Omega^{n,0}$

Let $K = \Lambda^{n,0} \Omega^M$, $n = C$-dim of $M$. 

Cor: If $M$ is closed & $E\to M$ is a holo line bundle, then $H^0(E) = 0$ & $q > 1$

Hence $H^0_x(M; E) = 0$ & $q > 1$

If $H^0_x(M; E) = H^0_x(M; K \otimes K^* \otimes E) = H^0_x(M; K^* \otimes E)$ then by the theorem this vanishes if $n+q > n+1$

There are weaker versions of Kodaira vanishing that are also useful. Let $(M, J, i)$ be a K-Herm, $(E, i)$ be a Herm holo line bundle. Then

Near each $z \in M$

we can pick a local orthonormal frame $\{X_j, \overline{X}_j, \ldots, X_n, \overline{X}_n\}$

with $J(X_j) = \overline{X}_j$, define $\alpha_j = \frac{1}{\xi_j} (X_j + i \overline{X}_j)$

so that $(\alpha_j, \overline{\alpha}_j, \ldots, \overline{\alpha}_n, \alpha_n)$ is a local orthonormal frame for $TM_0^\perp$

Denote its dual frame by $(\theta_j, \overline{\theta}_j, \ldots, \overline{\theta}_n, \theta_n)$.

Write $\mathcal{L}^\infty = \sum q_j \theta_j \otimes \overline{\theta}_j$

For a local section $\sigma$ of $E$

& a differential form $\alpha = \theta_j \otimes \overline{\theta}_j \otimes \sigma$

we have $[\theta_j \otimes \overline{\theta}_j \otimes \Lambda] \alpha = \begin{cases} \alpha & \text{if } j \in J^\perp \cup J' \cup J'' \\ -\alpha & \text{if } j \notin J^\perp \cup J' \cup J'' \\ 0 & \text{otherwise} \end{cases}$

It follows that $[(\mathcal{L}^\infty, \Lambda)] \alpha = \begin{cases} \sum q_j - \sum q_j \alpha & \text{if } j \in J^\perp \cup J' \cup J'' \\ 0 & \text{otherwise} \end{cases}$

This refines the case $[(\mathcal{L}, \Lambda)] \alpha = (q_1 - q_2 - n)$ which corresponds to $q_j = 1$ for

since $J^\perp \cup J' \cup J'' = J^\perp \cup J' \cup J'' \cup J'' \cup J'' \cup J'' - n = J + J' - n$
Then let $M$ be a closed $K$-Hausdorff manifold $M$ for $M$ in $E\rightarrow M$ a holomorphic bundle. Then metric $\omega$ be a correspondence $\Omega^\omega$ of $\Omega^\omega$ connection $\Omega^\omega$. Let $\Omega: M \rightarrow E\rightarrow \mathbb{R}$ satisfy

$$|\Omega^\omega(V, W) + 2i \omega(V, W)| \leq \varepsilon |V| |W|$$

for all $V, W$.

Let $H^0_\omega(M; E) = 0$ for all satisfying

$$3(n - |v - q|) \leq \kappa (p + q - n)$$