Math 514

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Last Time:

1) If $E \to M$ is a holomorphic vb over a closed Kähler mfd $M$, then a Hermitian metric $h$ induce a corr. Chern connection $\nabla_h$.

Then the Dolbeault desc, Frölicher desc, etc. hold for the cohomology of $\mathcal{O}(E)$, with $\Omega^p \cong \Omega^p$.

2) The cohomology class $[\mathcal{O}(E) \cap H^p(M; \mathcal{E})]$ is independent of $h$.

It obstructs the existence of a Hermitian metric with $\mathcal{D}^{\nabla_h} = 0$.

It is known as the Atiyah class of $E$, $\alpha(E)$.

Remark: A connection on $E$ is "compatible with the holomorphic" if $\nabla_h$ is $\nabla_e$. If furthermore, $\nabla_e$ sends holomorphic sections to holomorphic sections (locally) then $\nabla_h$ is said to be a "holo. conn."

Atiyah showed that $\alpha(E) = 0$ vanishes iff $E \to M$ admits a holo. conn.

3) For a complex line bundle $E \to M$ (holomorphic or not),

the curvature of a connection $\mathcal{D}$ defines a class in $H^2(M; \mathcal{E})$ independent of the choice of connection.

The $1^{st}$ Chern class of $E$ is $c_1(E) = \frac{1}{2\pi i} [\mathcal{O}(E) \cap H^2(M; \mathcal{E})]$.

If $E$ is a holo line bundle then the inclusion $H^1(M) \to H^1(M)$

sends $\frac{1}{2\pi i} \alpha(E)$ to $c_1(E)$

since they are represented by the same diff. form.
Rank $k$, for a $C$-vb $E \to M$ at rank $r$, $d^r R^o = 0$

so $[R^o] \in H^r(M; \text{End}(E))$

If $f : M_r(E) \to C$ is a polynomial function s.t. $f(S^r T) = f(T)$
then $f([R^o]) \in H^{2r}(M; C)$ ($= \oplus H^{2r}(M; C)$)

One can show that $d f([R^o]) = 0$

so that $[f([R^o])] \in H^{2r}(M; C)$ is independent of the choice of $f$

These are called the "characteristic classes of $E"$ (Chern-Weil const.)

Define $c_k(E)$ by $\det (I_T - t \frac{\nabla}{\nabla}) = \sum c_k w t^k$

thus $c_k(E) = [c_k([R^o])] \in H^{2k}(M; C)$ is the $k$th Chern class of $E$

Recall that $E \cong V^r \otimes \mathbb{C}$, $E = \phi^* V^r$

$\downarrow$

$M \to \text{Gr}_r$

In terms of $\phi$, $c_k(E) = \phi^* (c_k(V^r)) \cup (c_k(V^r))$ generate $H^{2k}(\text{Gr}_r)$

This time: Positive $d\Omega$ forms

We say that a (real) diff form $\omega$ of type $(1,1)$ is positive if $\omega(V\wedge J\omega) > 0$ is positive definite (i.e. a Riemannian metric)

Thus if $\omega$ is closed & positive the setting $g(V\wedge J\omega) = \omega(V\wedge J\omega)$

the mfd $(M, J, \omega, d\omega)$ is $K$-Hermitian
We say that a cohomology class \( c \in H^{1,1}(M; \mathbb{C}) \) is positive if it has a positive representative.

We say that a holomorphic line bundle \( E \rightarrow M \) is positive if its \( 1^{\text{st}} \) Chern class is positive.

Let's consider the holomorphic bundles over \( \mathbb{C}P^n \).

First, let's notice that tautological bundles have natural Hermitian metrics. Indeed, if \( E \rightarrow M \) is \( \pi^*(\mathbb{C}^n) \rightarrow \text{Gr}_r(\mathbb{C}^n) \).

Then, by definition \( \pi^*(\mathbb{C}^n) = \{ (W, v) \in \text{Gr}_r(\mathbb{C}^n) \times \mathbb{C}^n : v \in W \} \) is a sub-bundle of the trivial bundle \( \mathbb{C}^n \rightarrow \text{Gr}_r(\mathbb{C}^n) \).

The standard metric on \( \mathbb{C}^n \) induces a bundle metric on \( \pi^*(\mathbb{C}^n) \).

We restrict to a bundle metric on \( \pi^*(\mathbb{C}^n) \rightarrow \text{Gr}_r(\mathbb{C}^n) \).

The Chern connection on \( \pi^*(\mathbb{C}^n) \) is given by \( \omega \).

By the projection \( \pi^* \) onto \( \mathbb{C}^n \).

If \( s : U \rightarrow \mathbb{C}^n \) is a local section, then we can think of it as a map \( s : U \rightarrow \pi^*(\mathbb{C}^n) \).

\[ \nabla_{\bar{v}} s = \pi^*(d_s(v)) \]

For the tautological bundle over \( \mathbb{C}P^n \), \( \mathcal{O}_{\mathbb{C}P^n}(-1) \rightarrow \mathbb{C}P^n \), this construction gives a metric ker on \( U \).

\[ \text{ker} \frac{dz_1^*}{dz_2^*} \]
Let's spell this out.

On $\mathbb{CP}^n$, a standard atlas $U_d: \{(z_0, \ldots, z_n) \in \mathbb{CP}^n : z_d \neq 0\}$

has transition function $g_{ap} = \left(\frac{z_d}{z_a}\right)^* = \frac{z_d}{z_a}$

A section of $L$, $s: \mathbb{CP}^n \to L$

can be decomposed into maps $s_d: U_d \to \mathbb{C}$
such that $s = g_{ap} s_d$

A Hermitian metric on $L$ can similarly be decomposed into

$m_p: U_d \to \mathbb{R}^+$ s.t. $h_p = \frac{1}{g_{ap}} h_d$

for which

$|z|^2 h_p(p) = |s_a(p)|^2 h_d(p)$ if $p \in U_d$

This is well-defined since, if $p \in U_a \cap U_d$, then

$|s_p(p)|^2 h_p(p) = |g_{ap}|^2 |s_d(p)|^2 |g_{ap}|^{-2} h_d = |s_d(p)|^2 h_d(p)$

Now

$h_d(z_0, \ldots, z_n) = 12 \xi^{-2} \left(12z_0^2 \ldots + 12z_n^2\right) = \left(\frac{z_d}{z_a}\right)^2 (\ldots + 12z_i^2)$

satisfies $h_d(z^{\overline{3}}) = \left(\frac{z_d}{z_a}\right)^2 h_d(z^{\overline{3}}) = \frac{1}{g_{ap}} h_d(z^{\overline{3}})$

and hence defines a Hermitian metric on $L$

In particular, in the chart $\mathbb{C}^n \ni z \to (z_0, \ldots, z_n) \in U_d$

d we have $h_d(z) = 1 + z_d^2 = 1 + \bar{z}\bar{z}$

Recall that locally, the connection form of the Chern connection

associated to $h$ is $\Theta = h^{-1} \partial h$

d the curvature form is $\Omega = \partial \Theta$.
Here we have \( \Theta = \hbar^{-1} \Theta \) where 
\[
\Theta = \frac{1}{1 + \|z\|^2} \\
\gamma \text{ is written as locally } \gamma^i = \frac{dz \wedge d\bar{z}}{1 + \|z\|^2} = -\frac{dz \wedge d\bar{z}}{(1 + \|z\|^2)^2} \\
\]
This shows that the tautological line bundle \( \mathcal{L} \to \mathbb{P}^n \) is locally \( \Theta \) negative.

A metric on \( \mathcal{L} = \mathcal{O}(-1) \) induces metrics on \( \mathcal{O}(-k) = \mathcal{O}(-1)^{\otimes k} \) and \( \mathcal{O}(k) = \mathcal{O}(-1)^{k*} \) for \( k > 0 \).

The corresponding Chern connections have curvature on \( \mathcal{O}(k) \)
\[
\Theta_{ik} = \frac{k}{(1 + \|z\|^2)^2} \\
\]
Thus \( \Theta(k) \) is positive if \( k \) is positive.

Hence for \( \mathcal{O}(1) \) we have \( \Theta_{ik} = -\omega_{gs} \), the fundamental form of the Fubini-Study metric.