**Math 514**  
Nov 9, 2020

**Last Time: Lefschetz identities**

Our current theme is to understand how the three compatable structures of a K-Hermitian mfd \((M, g, J, \omega)\) interact with cohomology.

\(-g\rightarrow\text{Hodge}, J\rightarrow\text{Dolbeaut}, \omega\rightarrow\text{Lefschetz}\)

**This Time: Lefschetz decomp.**

The latter will involves

\(L: \Lambda^k(M) \rightarrow \Lambda^{k+1}(M), L\omega = \omega \wedge \omega, L(\Lambda^0) = \Lambda^{k+1}(M)\)

\(\Lambda: \Lambda^k(M) \rightarrow \Lambda^{k-1}(M), \Lambda(\Lambda^0) = \Lambda^{k-1}(M), \Lambda = - * L^*\)

\(\Pi: \Lambda^*(M) \rightarrow \Lambda^*(M), \Pi \omega = k \omega, \Pi \omega \in \Lambda^k(M)\)

We've shown that

i) \([L, \Lambda]: \Lambda^k(M) \rightarrow \Lambda^k(M), [L, \Lambda] \omega = (k-n+1) \Lambda \omega, k \in \Lambda^k(M)\)

ii) \([L, \Pi]: \Lambda^k(M) \rightarrow \Lambda^k(M), \Pi \omega = k \omega, \Pi \omega \in \Lambda^k(M)\)

iii) For \(k \leq n\), \(L^{k-n}: \Lambda^k(M) \rightarrow \Lambda^{2n-k}(M)\) is an isomorphism

\(\Pi: L^k: \Lambda^{k-n}(M) \rightarrow \Lambda^{2n-k}(M)\) is an isomorphism

**Let's say that a form \(\alpha\) is not primitive if it is in the image of \(L\), \(\alpha = \omega \wedge \omega\).**

It's not primitive because it "comes from" a lower degree form.

We can write \(\Lambda^*(M) = \text{Im} L \oplus \text{Ker} L^* = \text{Im} L \oplus \text{Ker} \Lambda\)

so we'll say that a form \(\alpha\) is primitive if \(\Lambda \alpha = 0\)

Lefschetz decomp builds up a form starting with primitive forms.
Lemma \( \alpha \in \Lambda^k(M), k < n \), is primitive iff \( L^{n-k+1} \alpha = 0 \)

**Proof**

We know \([L^{n-k+1}, \Lambda] \alpha = 0\), i.e., \(L^{n-k+1} \Lambda \alpha = \Lambda L^{n-k+1} \alpha\)

Now \(\Lambda \alpha \in L^{n-k}(M) \wedge L^{n-k}(M) \wedge \Lambda \alpha = 0\).

so \(L^{n-k+1}\) is injective on \(L^{n-k}(M)\), then \(L^{n-k+1} \alpha = 0 \Rightarrow \Lambda \alpha = 0\).

Also \(L^{n-k+1} \alpha \in L^{2n-k+2}(M) \wedge L^{k-2} \alpha = 0\).

so \(\Lambda \alpha \) is injective on \(L^{2n-k+2}(M)\) \(\alpha\), then \(\Lambda L^{n-k+1} \alpha = 0 \Rightarrow \Lambda^{n-k+1} \alpha = 0\).

Hence a primitive \(\Rightarrow \Lambda \alpha = 0 \Rightarrow L^{n-k+1} \alpha = 0\).

**Prop.** (Lefschetz decomposition of diff forms)

Every \(\alpha \in \Lambda^k(M)\) admits a unique decomposition of the form

\[\alpha = \Sigma L^r \alpha_r \quad \forall \alpha_r \text{ of degree } k-2r \leq \min(2n-k, k)\]

and primitive.

**Proof**

**WLOG assume** \(k < n\)

Let's start with uniqueness.

Suppose \(\Sigma r \geq 0 L^r \alpha_r = 0\) want to show \(\alpha_0 = 0\)

If \(\alpha_0 = 0\) then \(L(\Sigma L^{-1} \alpha_r) = 0 \Rightarrow \Sigma L^{-1} \alpha_r = 0\)

so inductively we may assume \(\alpha_0 \neq 0\).

Now \(\alpha_0 \in \Lambda^k(M) \wedge \text{primitive} \Rightarrow \text{we know } L^{n-k+1} \alpha_0 = 0\)

but \(\text{then } L^{n-k+1}(\Sigma L^{-1} \alpha_r) = 0 = L^{n-k+1} (\Sigma \text{degree } \alpha_r)\)

\& \(L^{n-k+2}\) is an isomorphism on \(L^{n-k}\) so \(\Sigma \geq 0 L^{-1} \alpha_r = 0\)

Induction on \(k \Rightarrow \alpha_r = 0 \forall r > 0\) & hence \(\alpha_0 = 0\).
Next let’s prove existence

\[ L^{-k+1} \alpha \in \mathfrak{L}^{-k+2}(M) = L^{-k+2}(\mathfrak{L}^{-k}(M)) \]

where \( \exists \mu \in \mathfrak{L}^{-k}(M) \) s.t. \( L^{-k+1} \alpha = L^{-k+2} \mu \)

so \( \alpha_0 = \alpha - L \mu \) is primitive & \( \alpha = \alpha_0 + L \mu \)

Induction on form degree allows us to assume that \( \mu \) has a
Lefschetz decomposition & hence so does \( \alpha_0 \).

**Remark:** If \( \alpha = \sum L^r \alpha_r \) & we complexify then

\[ T_{0,2} \alpha = \sum L^r T_{r,-1,2} \alpha_r \]

The Lefschetz decomposition descends to cohomology.

**Lemma:** On a K-Hermitian manifold, \([\Delta_0, L] = 0\).

**Proof:**

We know that \([\omega, L] \alpha = \partial(\omega \wedge \alpha) - \omega \wedge d \alpha = 0\)

& \([\omega^*, L] = -i\omega^*\]

hence \([\Delta_0, L] = 2[\Delta_0, L] = 2 (2[\omega^*, L] + [\omega^*, L]^2)\]

\[ = 2 (2[\omega^*, L] + [\omega^*, L]^2) = -2i (\omega^* + \overline{\omega}^*) = 0\]

**Theorem (Hard Lefschetz Theorem):** \((M, g, J, \omega)\) closed K-Hermitian of C-dim \( n \)

\([\Delta_0, L] = 0 \) for all \( k \in \mathbb{N} \), \( L^{-k} \text{ induces an isomorphism } H^k(M) \rightarrow H^k(M) \)

**Proof:** Let \( H^k(M) = \ker \Delta_0 | L^{-k} \cong H^k_{\mathrm{dR}}(M) \)

Since \([\Delta_0, L] = 0 \), \( L^{-k} : H^k(M) \rightarrow H^{2n-k}(M) \) is by above an injective

\( \mathbb{C} \) linear, \( \dim H^k(M) = \dim H^{2n-k}(M) \) since it is an isomorphism

& so \( L^{-k} \) is also surjective.
Cor (Lefschetz decomposition of cohomology)
If $H^k(M)_{prim} = \ker L^{n-k+1} \leq H^k(M)$ (for $k \leq n$)
Ker, for any $k$, $H^k(M) = \oplus L^k H^{k-2n}(M)_{prim}$

Remark: In particular, if $k \leq n$, $b_k \leq b_{k-2}$ ($h^{r,2} \leq h^{0,2(r+1)}$)
& if $k > n$, $b_k \geq b_{k-2}$ ($h^{p,2} \geq h^{r,2(r+1)}$)
Indeed, $\dim H^k(M)_{prim} = b_k - b_{k-2}$