Last Time: Hodge cohomology, \( H^k_\omega(M) = \ker \Delta_k \cong H^k_\Delta(M) \)

This Time: Poincaré Duality

If \( M \) is closed & orientable its cohomology satisfies:

Poincaré Duality, \( H^k_\omega(M) \cong H^{m-k}_\Delta(M) \) \( \forall k \) \( m = \text{real dim of } M \)

In fact there's an operator \( \ast : \Lambda^k(M) \to \Lambda^{m-k}(M) \) \( \forall k \)

known as the "Hodge star" s.t. \( \ast \Delta_k = \Delta_{m-k} \ast \)

It descends to an isomorphism \( \ast : H^k_\omega(M) \cong H^{m-k}_\Delta(M) \)

(This is true for any choice of Riem. metric \( g \))

We define \( \ast \) by requiring that for any \( \alpha, \beta \in \Lambda^k(M) \)

\[ \alpha \wedge \ast \beta = g(\alpha, \beta) \, \text{d}V_3 \]

E.g., on \( \mathbb{R}^3 \) with \( \text{d}V_3 = \text{d}x_1 \wedge \text{d}x_2 \wedge \text{d}x_3 \)

so \( \ast 1 = \text{d}x_1 \wedge \text{d}x_2 \wedge \text{d}x_3 \)

\[ \ast \text{d}x_1 = \text{d}x_2 \wedge \text{d}x_3, \quad \ast \text{d}x_2 = -\text{d}x_1 \wedge \text{d}x_3, \quad \ast \text{d}x_3 = \text{d}x_1 \wedge \text{d}x_2 \]

\[ \ast (\text{d}x_1 \wedge \text{d}x_2) = \text{d}x_3, \quad \ast (\text{d}x_1 \wedge \text{d}x_3) = -\text{d}x_2, \quad \ast (\text{d}x_2 \wedge \text{d}x_3) = \text{d}x_1 \]

\[ \ast \text{d}x_1 \wedge \text{d}x_2 \wedge \text{d}x_3 = 1 \]

(For a general coord chart \( \text{d}V_3 = \sqrt{|\det g|} \, \text{d}x_1 \wedge \ldots \wedge \text{d}x_n \))

\[ \ast^2 = \pm \text{id} \] in fact \( \ast^2 = (-1)^{k(m-k)} \) on \( k \)-forms

The \( L^2 \)-pairing on \( \Lambda^k(M) \) is then \( \langle \alpha, \beta \rangle_M = \int_M \alpha \wedge \ast \beta \)
We can express $\delta = d^* \alpha$ in terms of $d \delta = d \Delta +$

$$\langle d\alpha, \beta \rangle = \langle \alpha, d\beta \rangle = \int_M d^* \alpha \wedge d\beta$$

$$= \int_M (\alpha \wedge \ast d\beta) = \ast \alpha \wedge d\beta = \int_M \alpha \wedge d\ast \beta$$

$$\Rightarrow \delta \beta = \pm d\ast \beta$$

Working out the signs we find that

$$\delta \alpha^m = (-1)^{m+1} \alpha \ast = d* \quad \text{(if } m \text{ is even then } \delta = -d\ast)$$

Hence

$$\Delta \alpha^m = \ast (d\delta + \delta d) \ast = (d\ast \Delta \ast + \ast \Delta d) \ast$$

$$= \pm \ast \Delta \ast \ast = \ast \Delta \ast = \pm \ast d\ast \ast \ast = \pm \ast d\ast \ast \ast$$

$$= \ast \Delta \ast$$

as required.

Suppose now that $M$ is a complex manifold with a Hermitian metric $h$ (let $m = 2n$ be the real dim of $M$, so $n$ is the complex dim)

Extend J to differential forms, $\omega \in \Omega^k(M) \Rightarrow J\omega \in \Omega^k(M)$

$(J\omega)(V_1, \ldots, V_k) = \omega(JV_1, \ldots, JV_k)$

(After complexifying this means that $J$ acts by multiplication by $i^{p-q}$ on $\Omega^{p,q}(M)$)

Define another $\ast$ order operator by

$$D = J^{-1} \circ d \circ J : \Omega^k(M) \to \Omega^{k+1}(M)$$

(In polar coordinates on $\mathbb{C}$, $z = re^{i\theta}$, we have

$$d = \partial_r \text{ ext} (dr) + \partial_\theta \text{ ext} (d\theta)$$

$$D^r = r \partial_r \text{ ext} (d\theta) + \frac{1}{r} \partial_\theta \text{ ext} (dr)$$

$$\Delta \ast dD^r = \left( \partial_x^2 + \partial_y^2 \right) \text{ ext} (dx \wedge dy)$$)
After complexifying, if \( \omega \in \Lambda^{\nu,\nu}(M) \)

\[
\text{then } d^{c} \omega = J^{-1} (\theta + \bar{\theta}) J \omega = J^{-1} (\theta + \bar{\theta}) \omega
\]

\[
= \left[ 1^{\nu,\nu} (\frac{1}{2} \omega + \frac{1}{2} \omega) \right] + i \left[ 1^{\nu,\nu} (\frac{1}{2} \omega - \frac{1}{2} \omega) \right]
\]

\[
= \frac{1}{i} \theta \omega + i \bar{\theta} \omega = i (\theta - \bar{\theta}) \omega
\]

In particular \( d^{c} I = (\theta - \bar{\theta}) + (\bar{\theta} - \theta) = 2i \theta \bar{\theta} = -d^{c} I \).

On the other hand, \( (d^{c})^{2} = J^{-1} d J J^{-1} d J = 0 \)

so we have a complex \( 0 \xrightarrow{d^{c}} \Lambda^{0,0}(M) \xrightarrow{d^{c}} \Lambda^{1,1}(M) \xrightarrow{d^{c}} \cdots \Lambda^{m,m}(M) \rightarrow 0 \)

A associated cohomology \( H_{k}^{c}(M) = \frac{\ker d^{c} | \Lambda^{k}}{\text{Image } d^{c} | \Lambda^{k-1}} \)

The (formal) adjoint of \( d^{c} \) is \(-d^{c} \).

Next, let's extend \( \ast \) to complexified differential forms

by requiring \( \omega \ast \bar{\eta} = h(\omega, \eta) \text{ d}V \)

If \( \omega = \sum u_{\alpha} \text{ d}z^{\alpha} \text{ d}\bar{z}^{\alpha} \) both have type \((p, q)\)

\[ v = \sum v_{\alpha} \text{ d}z^{\alpha} \text{ d}\bar{z}^{\alpha} \]

then \( h(\omega, v) = \sum u_{\alpha} \overline{v_{\alpha}} \) &

\[ u \ast v = h(\omega, v) \text{ d}z_{1} \text{ d}\bar{z}_{1} \cdots \text{d}z_{n} \text{ d}\bar{z}_{n} \]

(for an orthonormal frame)

In particular since \( \text{d}V \) has type \((n,n)\)

\( \ast \) is a \(C\)-linear isometry \( \Lambda^{0,\nu}(M) \rightarrow \Lambda^{n-\nu, n-\nu}(M) \)

Note that the decomposition \( \Lambda^{n}(M) = \bigoplus_{\nu=0}^{n} \Lambda^{\nu, \nu}(M) \)

is orthogonal with respect to the \( L^{2} \)-product.
The formal adjoints of $\overline{\theta}$ are $\overline{\theta}^* : L^{n+1}_0(M) \to L^{n-1,0}_0(M), \overline{\psi}^* : L^{n+1}_\psi(M) \to L^{n-1,\psi}_0(M)$

Claim: $\overline{\theta}^* = -\ast \overline{\psi}^*, \overline{\psi}^* = -\ast \overline{\theta}^*$

\[
\langle \overline{\theta} \omega, \eta \rangle_{\ast - \ast} = \langle \omega, \overline{\theta}^* \eta \rangle_{\ast - \ast} \quad \text{for } \omega \in L^{n-1,0}_0(M), \eta \in L^{n+1}_0(M)
\]

\[
= \int \omega \ast \overline{\psi} \eta \\
= \int \overline{\psi} (\omega \ast \overline{\eta}) \ast \omega \ast \overline{\psi} \eta = \int \overline{\psi} (\omega \ast \overline{\eta}) \ast \omega \ast \overline{\psi} \eta = \int \omega \ast \overline{\eta} \overline{\theta} \ast \eta \\
= \int \omega \ast \ast \overline{\theta} \ast \eta = \int \omega \ast \ast \overline{\psi} \ast \eta \quad \text{(hence } \overline{\theta} = -\ast \overline{\psi}^*)
\]

Taking conjugates, $\overline{\psi}^* = -\ast \overline{\theta}^*$

Define $\Delta_1 = d d^* + d^* d, \Delta_{\overline{\psi}} = d^\dagger d + d d^\dagger$

$\Delta_\theta = \theta \overline{\theta}^* + \overline{\theta} \theta^*, \Delta_{\overline{\psi}} = \overline{\psi} \overline{\psi}^* + \overline{\psi}^* \overline{\psi}$

We'll see that $\sigma(\Delta_1)(\mathcal{E}) = 11 \frac{1}{2} \lambda^2 = \sigma(\Delta_{\overline{\psi}}) = 2 \sigma(\Delta_\theta) = 2 \sigma(\Delta_{\overline{\psi}})$

Hence these are all elliptic & we elliptic theory tells us that, on a closed manifold, they have f.d. nullspaces