Math 514

Last Time: Local characterizations of K-Hermitian structure

$(M, g, J, \omega)$ K-Herm iff $\nabla J = 0$ or $\nabla \omega = 0$

This Time: Elliptic operators

First, let's set up the adjoint of a differential operator.

If $M$ is a smooth manifold, a linear differential operator of order $k$

$L : \text{Diff}^k(M) \to \text{C}^0(M, \mathbb{F})$

that for any choice of local coordinates takes the form

$L f = \sum_{\alpha} a_\alpha(\xi) D^\alpha f = \sum_{\alpha_1, \ldots, \alpha_n} a_{\alpha_1, \ldots, \alpha_n}(\xi) \partial_1^{\alpha_1} \partial_2^{\alpha_2} \ldots \partial_n^{\alpha_n} f$

so $L$ is a polynomial in vector fields.

In fancier language, $L$ is an element of the enveloping algebra of $\mathfrak{X}(M)$
or equivalently by Petree's the $L$ is a linear map that does
not increase support, i.e., supp $Lf \subseteq \text{supp } f$
or Grothendieck's approach is to define $\text{Diff}^k(M)$ inductively

with respect to $L$. $\text{Diff}^k(M)$ is multiplication by a smooth kernel

$L : \text{Diff}^k(M) \to \text{C}^0(M, \mathbb{F})$

If $E \to M$, $F \to M$ are vb's over $M$, then $L : \text{Diff}^k(M; E, F) \to \text{C}^0(M; E, F)$

is a linear map $L : \text{C}^0(M; E) \to \text{C}^0(M; F)$

that in local coordinates has the form above $\omega_\alpha(\xi) \in \text{Hom}(E_\xi, F_\xi)$

The expression for $L$ in local coordinates depends strongly on
the choice of coordinates, but the "top order part" can
be intrinsically defined $\Delta$ is called the "principal symbol" of $L$. 
If \( \text{Diff}^k(M; E; F) \), its principal symbol \( \sigma_k(L) \) is the map
\[
T^*_x M \to \text{Hom} (E_x, F_x)
\]
\[
\xi \mapsto \sum_{|\alpha| = k} \xi^\alpha \sigma_k (\xi) \partial_x^\alpha (i \xi)^\alpha
\]
obtained from the \( k \)-th order derivative by replacing \( \partial_x^\alpha \) with \( i \xi^\alpha \).

The motivation comes from Fourier transform.

If \( M = \mathbb{R}^n \) and \( \hat{\phi}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} \phi(x) \, dx \), \( \phi \in C_c^\infty(\mathbb{R}^n) \)
then \( \hat{\partial_x^j \phi}(\xi) = i \xi^j \hat{\phi}(\xi) \).

So, for any constant coefficient differential \( L \) on \( \mathbb{R}^n \)
we have \( \hat{\sigma(L \phi)}(\xi) = \hat{\phi}^{(L)}(\xi) \hat{\phi}(\xi) \)
for some polynomial \( \hat{\phi}^{(L)} \) (called the full symbol of \( L \)).

Let its homogeneous part of degree \( k \) be the principal symbol \( \sigma_k(L) \).

E.g., \( \Delta = -\sum \partial_{x_j}^2 \) satisfies \( \hat{\sigma(\Delta \phi)} = -\sum (ix_j)^2 \hat{\phi}(\xi) = |\xi|^2 \hat{\phi}(\xi) \)
so, \( \sigma_k(\Delta)(\xi) = |\xi|^k \).

How do we know it's well-defined?

Say \( \text{Diff}^k(M; E; F) \) is as above \( \xi \in T^*_x M \)
Pick \( \phi \in C_c^\infty(M) \) s.t. \( \Delta \phi = \xi \)
and note that \( \sigma_k(L)(\xi) = \lim_{t \to 0} \frac{e^{-it \phi} \Delta (e^{it \phi})}{t^k} \)

Indeed, \( \partial_{x_j} e^{it \phi} = (i + \partial_{x_j} \phi) e^{it \phi} \)
\( \partial_{x_j}^2 e^{it \phi} = (i + \partial_{x_j} \phi)^2 e^{it \phi} + \text{(lower order terms in } t \text{)} e^{it \phi} \)
so \( e^{it \phi} \Delta (e^{it \phi}) = t^k \sigma_k(L) + \text{(lower order terms in } t \text{)} \)
The principal symbol is local in $\mathcal{X}^k$, if $\mathcal{X} \in \mathcal{C}^\infty(M)$,
then $\sigma_k(\mathcal{X}L)(\xi) = \mathcal{X}(\xi) \sigma_k(L)(\xi)$ for $\xi \in T^*_xM$
hence $\sigma_k(L) \in \mathcal{C}^\infty(T^*M; \Pi^* \text{hom}(E,F)) \forall \Pi: T^*M \to M$

An operator $L$ is called elliptic if $\sigma_k(L)(\xi)$ is invertible
$\forall \xi \in T^*_xM \setminus \{0\}$, $x \in M$

eg: $\Delta$ on $\mathbb{R}^n$, $\sigma_\nu(\Delta)(\xi) = |\xi|^2$

Next, let’s check that the (formal) adjoint of an elliptic diff operator is again elliptic.

On a Riemannian mfld $(M,g)$
Here’s on $L^2$-pairing on $\mathcal{C}^\infty(M)$:
$\mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(M)$,
\[ (f_1, f_2) \rightarrow \int_M f_1 f_2 \, dv_g \]
\[ \|f\|_{L^2} = (f, f)_{L^2}^{1/2} = \left( \int_M f^2 \, dv_g \right)^{1/2} \]
$L^2(M)$ is the completion of $\mathcal{C}^\infty(M)$ w.r.t. $\|\cdot\|_{L^2}.$

If $E$ is an $F$-vb over $M$ equipped with $F$-bi$$k$$ metric $h^E$, then there’s on $L^2$-pairing on $\mathcal{C}^\infty(M; E)$:
\[ \mathcal{C}^\infty(M; E) \rightarrow \mathcal{C}^\infty(M; E) \]
\[ (s, s) \rightarrow \int_M h^E(s, s) \, dv_g \]
which yields $L^2(M; E)$ ($= L^2(M, g; E, h^E)$).
If $L \in \mathcal{D}(\mathcal{H}^k(M, E; F))$ & we equip $M$ with a metric $g$

& $E, F \rhd$ $\mathcal{D}(\mathcal{H}^k(M, E; F))$

Then the (formal) adjoint of $L$ is the operator $L^* \in \mathcal{D}(\mathcal{H}^k(M; F, E))$

determined by

$$(L^* \zeta, \tilde{\xi})_E = (\zeta, L^* \tilde{\xi})_F \n \forall \zeta \in C_c(M; \mathbb{C}), \tilde{\xi} \in C_c(M; \mathbb{C}; F)$$

(i) If $M$ is closed then $C_c(M; \mathbb{C}) = C^\infty(M; \mathbb{C})$

The principal symbol of $L^*$ satisfies

$h^E(\sigma LL)(\xi) u, v) = h(u, \sigma L^*(\xi) v) \n \forall \xi \in \mathcal{T}'M, u \in E, v \in F$

$h_L(\xi) = \sigma(L^*)^\xi$

In particular, $L$ is elliptic if & only if $L^*$ is elliptic.

Also note that the principal symbol is a homomorphism,

i.e. $\sigma(L \circ L^*) = \sigma(L) \circ \sigma(L^*)$

Then $M$ closed smooth manifold, $E, F \rhd M, \forall u, v, L \in \mathcal{D}(\mathcal{H}^k(M, E; F))$

If $L$ is elliptic then:

i) $\ker L = \ker \sigma L = \{u \in C^\infty(M; E) : Lu = 0\}$ is finite dimensional

ii) $\text{Image } L = L(C^\infty(M; E))$ is a closed subspace of $C^\infty(M; F)$

iii) $\text{Coker } L = C^\infty(M; E) / L(C^\infty(M; E)) \cong \ker L^*$ is finite dimensional

Thus $C^\infty(M; E) \cong \ker L \oplus \text{Image } L^*$

$C^\infty(M; F) \cong \ker L^* \oplus \text{Image } L$

This is also true if we replace all instances of $C^\infty(M; E)$

Moreover $\ker \sigma L = \ker \sigma L^*$ (elliptic regularity).