Math 514

Oct 14, 2020

Last Time: Chern connections

This Time: K-Hermitian manifolds

A Hermitian manifold is a complex manifold $M$ together with

1. a Riemannian metric $g$
2. an integrable almost $C$-structure $J$
3. a two form $\omega$ (real-valued)

such that $g(J(u),J(v)) = g(u,v)$ & $\omega(u,v) = g(J(u),v)$

If $(M, g, J, \omega)$ is a Hermitian mfd then

$h(u,v) = g(u,v) - i\omega(u,v)$ is a Hermitian metric on $(TM, J)$

A complex mfd $M$ is Kähler (or K-Hermitian) if

it is Hermitian $\Delta \omega = 0$

In which case $\omega$ is a symplectic form

Let's describe a Hermitian structure in local holomorphic words $\{\xi_j\}$

Let $H \in \text{GL}_n(C)$ be the matrix with entries $h_{jk} = h(\partial \xi_j, \partial \xi_k)$

Then $H = H^t$ & positive definite

Recall that the natural $C$-vb isomorphism $(TM, J) \xrightarrow{i} T^* \otimes M$

$\xi(v) \mapsto \frac{1}{2}(v - iJ(v))$

To find the Riemann metric write $\xi_j = x_j + iy_j$

& note that $\xi_j(\partial x_j) = \partial \xi_j$ & $\xi_j(\partial y_j) = \xi(J(\partial x_j)) = i\partial \xi_j$

Thus we have, for instance, $g(\partial x_j, \partial x_k) = \Re h(\partial \xi_j, \partial \xi_k) = \Re h_{jk}$

& $g(\partial y_j, \partial y_k) = \Re h(\partial y_j, \partial y_k) = \Re (i\partial \xi_j, \partial \xi_k) = \Im h_{jk}$

So, in the basis $\partial x_1, ..., \partial x_n, \partial y_1, ..., \partial y_n$, $g$ is the $2n \times 2n$ matrix $G = \begin{pmatrix} \Re H & \Im H \\ -\Im H & \Re H \end{pmatrix}$
Next let $\omega$ consider the 2-form $\omega$

$$\omega(x_1, x_n) = -\text{Im} \ h(z_j, z_n) = -\text{Im} \ h_{jn}$$

$\omega(x_j, x_n) = -\text{Im} \ h(z_j, z_n) = -\text{Im} \ h_{jn}$

$\omega(y_j, y_n) = -\text{Im} \ h(z_j, z_n) = -\text{Im} \ h_{jn}$

This looks nicer if we view $\omega$ as a $\mathbb{C}$-valued 2-form

by extending it bilinearly to the complexified target space $TM \otimes \mathbb{C}$
(here we have to carefully distinguish between $J$ & $i$).

We want to express $\omega$ in terms of $dz_j \wedge d\overline{z}_n$

$$\omega(z_j, \overline{z}_n) = \omega(z_j, \overline{z}_n) + \text{Im} h_{jn} + i \text{Im} h_{jn} = \text{Im} h_{jn} = 2i \text{Im} h_{jn}$$

Similar computations show that $\omega(z_j, \overline{z}_n) = 0$, $\omega(z_j, \overline{z}_n) = 0$

so $\omega = \frac{i}{2} \text{Im} h_{jn} dz_j \wedge d\overline{z}_n$

In particular note that $\omega$ is of type $(1,1)$

$S^2(n) \cap S^{11} = \text{so we say $\omega$ is a real form of type (1,1)}$

**Example:** $\mathbb{C}^n$ with the standard metric so that $z_1, \ldots, z_n$ is a unitary basis

$H = \text{Id}_n$, $G = \left( \begin{array}{cc} i & 0 \\ 0 & i \end{array} \right) = \text{std metric on } \mathbb{R}^n$

$\omega = \frac{i}{2} \sum dz_j \wedge d\overline{z}_j = \sum dx_j \wedge dy_j = \text{std symplectic form on } \mathbb{R}^n$

Note $dw = 0$ so this is K-Hermitian

**Ex:** The K-Herm structure on $\mathbb{C}^n$ is translation invariant so it descends to a K-Herm str on $\mathbb{C}$-tori

**Ex:** Any Herm str on a Riemann surface is automatically K-Herm.
Ex: If NSM is a complex submanifold of a Hermitian structure on M restricted to a Hermitian structure on N, which is K-Hermitian if the one on M is.

Ex: CP^n admits a (U(n+1)-inv) K-Hermitian metric, known as the Fubini–Study metric.

Let z be std coordinates on C^m+1 & let ρ = |z|^2 = z \bar{z}. Set

\[ \omega = \frac{i}{m} \partial \bar{\partial} \log \rho = \frac{i}{m} \left[ \frac{\partial \bar{\partial} \log \rho}{\rho} \right] \]

\[ = \frac{i}{2 \pi} \left[ \frac{1}{|z|^4} \sum \frac{\partial z_i \partial \bar{z}_j}{z_i \bar{z}_j} - \left( \sum \frac{\partial z_i \partial \bar{z}_j}{z_i \bar{z}_j} \right)^2 \right] \]

U(n+1)-inv since it only depends on ρ
\& C^\infty since the numerator & denominator are homogeneous of deg 4.
Hence ω pulls back to a 2-form on CP^n.
To see that it is positive-definite (meaning ω(\cdot,\cdot) > 0)
evaluate it at the point (1:0:0:...:0) where it is clearly > 0.
A then appeal to U(n+1)-invariance to see that it is positive at all points.

Let (M,h) be a C^\infty manifold with a Hermitian metric h.
We can always find a local unitary frame for h
i.e., smooth sections of T^*_h M whose values give a unitary basis at each point.
For such a frame Z_1, ..., Z_n, we would have h(Z_j, Z_k) = \delta_{jk}.
Indeed, start with any frame & apply Gram–Schmidt.
If \( z_1, \ldots, z_n \) is a local unitary frame

A \( \Theta_1, \ldots, \Theta_n \) is a dual coframe

(i.e., \( \Theta_j \) are determined by \( \Theta_j(z_k) = \delta_{jk} \))

Then we have \( w = \frac{1}{n!} \sum_j \Theta_j \wedge \overline{\Theta}_j \)

And so \( w^n = w \wedge \cdots \wedge w = n! \frac{1}{n!} (\Theta_1 \wedge \overline{\Theta}_1 \wedge \cdots \wedge \Theta_n \wedge \overline{\Theta}_n) = n! \, d\omega^n \)

In particular, if \( d\omega = 0 \) & \( M \) is compact

Then \( w \) is not exact & neither is \( w^n \) & \( k \in \mathbb{N} \)

Indeed, if \( w \) were exact then so too would \( w^n \) be exact

(If \( \alpha \) is closed & \( p \) exact then \( d\alpha \wedge p \) is exact)

But if \( w^n \) were exact then Stokes then would say \( \int_M w^n = 0 \)

So knowing that \( \int_M w^n = n! \, Vol(M) \neq 0 \)

tells us that \( w^n \) is not exact; so \( w \) is not exact

Thus \( [w^n] \in H^{2k}_d (M) \) are not zero for all \( 1 \leq k \leq n \)