Math 514

LAST TIME:

1) Every holo vb $E \to M$ has a canonical diffr operator

$$\delta^E : \mathcal{O}^0(M; E) \to \mathcal{O}^0(M; E) \quad \nabla(\delta^E) = 0$$

which in turn determines the holomorphic structure of $E \to M$

2) A Hermitian metric on $E \to M$ is a smooth family of Hermitian inner products on the fibers of $E$

(Our convention is that $h^E$ is $C$-linear in the 1st entry

& conjugate linear in the 2nd entry. Often the convention of $C$-linearity in both entries is used.)

$$h^E \text{ is equiv to a $C$-anti-linear bdl isomorphism } \varphi^E : E \to E^*$$

$$(\varphi^E)(v)(w) = h^E(v, w)$$

3) Every $C$-vb admits a Hermitian metric

THIS TIME: Compatible metrics & connections

We say that a covariant derivative $\nabla^E$ & a Hermitian metric $h^E$

are compatible if, for any sections $s, s' \in \mathcal{C}^0(M; E)$ we have

$$\delta^E (h^E(s, s')) = h^E(\nabla^E s, s') + h^E(s, \nabla^E s')$$

&

$$\nabla^E (h_E(s, s')) = h_E(\nabla^E s, s') + h_E(s, \nabla^E s') \quad \forall \in \mathcal{C}^0(M; TM)$$
Thus if $M$ is a complex manifold and $E \to M$ a hermitian vector bundle, then for every hermitian metric $h^E$ on $E$ there is a unique hermitian connection $\nabla^E$ on $E$ compatible with $h^E$.

$(\nabla^E)$ is called the Chern connection of $(E, h^E)$.

Recall $\nabla^E$ is a hermitian connection if $(\nabla^E)^{0,1} = \overline{\partial}^E$.

If $E \to M$ is holomorphic then so is $E^* \to M$.

We give a connection $\nabla^E$ on $E$ we obtain one on $E^*$ by demanding $d(\omega(s)) = (\nabla^E \omega)(s) + \omega(\nabla^E s)$ for $\omega \in C^0(M; \mathfrak{V})$, $s \in C^0(M; E)$.

In particular, if $\nabla^E$ is a holomorphic connection on $E$ then $\nabla^E$ is a holomorphic connection on $E^*$.

It sends holomorphic sections to $(1,0)$-forms.

An equivalent way of expressing the compatibility of $\nabla^E$ is $\nabla^{E^*}$ is to say that $h^E(\nabla{E^*}s) = \nabla^{E^*} h^E(s)$ for $s \in C^0(M; E)$.

but note that the $C^*$-anti-linearity of $h^E$ implies that whenever $\omega \in C^0(M; T^*M \otimes E)$ we have $h^E(\nabla^E \omega(s)) = \nabla^{E^*} h^E(s)$.

hence $(\nabla^{E^*})^{1,0} s = (h^E)^{-1} (\overline{\partial}^E)^{0,1} h^E(s) = (h^E)^{-1} (\overline{\partial}^E h^E(s))$.

Thus a holomorphic connection compatible with $h^E$ must be equal to

$$\nabla^E = (h^E)^{-1} \overline{\partial}^E h^E + \overline{\partial}^E$$

Conversely, this formula gives existence.

Remark: It follows from the formula for the Chern connection that $\nabla^E$ has type $(1,1)$. Indeed the $(0,1)$-part vanishes since $(\overline{\partial}^E)^{0,1} = 0$ and the $(2,0)$-part vanishes since $(\overline{\partial}^E)^{2,0} = 0$. 
On a complex manifold, this discussion applies to the holomorphic tangent bundle $T_0 M$. We want to understand how it relates to the underlying smooth structure.

**Def** A Riemannian metric on a smooth manifold $M$ is a smoothly varying family of positive definite inner products on the fibers of the tangent bundle. That is, for each $x \in M$, a map $g_x : T_x M \times T_x M \to \mathbb{R}$ satisfying:

i) $g_x(u, v)$ is $\mathbb{R}$-linear in $u$ for all $v$

ii) $g_u(v, v) > 0$ for all $v 
eq 0$

iii) $g_x(u, v) = 0$ if and only if $u = 0$

iv) if $u, v \in C^\infty(M, TM)$ then $g(u, v) \in C^\infty(M)$

If $(M, \mathbf{J})$ is a almost $\mathbb{C}$-manifold & $h = h^\mathbb{T}M$ is a Hermitian metric on TM (viewed as a $\mathbb{C}$-vb).

Separate $h$ into real & imaginary parts $h(u, v) = g(u, v) + i\omega(u, v)$

Then $g$ is a Riemannian metric on $M$ & $\omega$ is a two-form, we $\mathbb{L}^2(M)$

Since $h(\mathbf{J}(u), \mathbf{J}(v)) = i(-i) h(u, v) = h(u, v)$

we have $g(\mathbf{J}(u), \mathbf{J}(v)) = g(u, v)$ & $\omega(\mathbf{J}(u), \mathbf{J}(v)) = \omega(u, v)$

Similarly $h(\mathbf{J}(u), v) = i\omega(u, v) = g(\mathbf{J}(u), v) = \omega(u, v)$ & $\omega(\mathbf{J}(u), v) = -\omega(u, v)$

In particular, having a Hermitian metric on TM as a $\mathbb{C}$-vb is equivalent to having a Riemannian metric on $\mathbb{T}M$

is compatible with $\mathbf{J}$ in that $g(\mathbf{J}(u), \mathbf{J}(v)) = \omega(u, v)$.

In fact, if $(g, \mathbf{J}, \omega)$ are compatible then any two determine the third. We sometimes refer to $(g, \mathbf{J}, \omega)$ as a Hermitian structure on $M$. 

Since $g$ is non-degenerate, so is $\omega$

The volume form of $g$ is equal to $\omega^n$

In particular $\text{Vol}(M, g) = \int_M \omega^n$

**Definition:** A Kähler manifold is a Hermitian manifold in which $d\omega = 0$

$\omega$ is called the Kähler form and $[\omega] \in H^2_{\text{cl}}(M)$ the Kähler class.

A non-degenerate closed 2-form is called a symplectic form.

Thus a Kähler manifold is a smooth manifold $M$

together with $(g, J, \omega)$ a compatible choice of

Riemannian metric, $J$-structure, and symplectic form

$K$-manifold, $K$-form, $K$-class