If \((M,J)\) is a manifold with almost complex structure

\[ \text{then } T^1 M \otimes \mathbb{C} = \overline{T^1 M} \oplus T^1 M \]

\[ J = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \]

\(J\) induces a bundle map \(T^* M \to T^* M\) by \(J(\omega)(v) = \omega(J(v))\)

A decomposition \(T^* M \otimes \mathbb{C} = T^* M^{1,0} \oplus T^* M^{0,1}\)

In local coordinates \(x_1, y_1, \ldots, x_n, y_n\) on \(M\)

\[ T^* M^{1,0} \text{ is spanned by } \theta_j = \frac{1}{2}(\partial x_j - i \partial y_j) \]

\[ T^* M^{0,1} \text{ is spanned by } d\bar{z}_j = dx_j + i dy_j \]

\& the \(T^* M^{1,0}\) is spanned by \(d\bar{z}_j = dx_j + i dy_j\)

\& \(T^* M^{0,1}\) is spanned by \(dz_j = dx_j - i dy_j\)

Sign work out so that

\[ dz_j(\theta_k) = \delta_{jk} \quad \text{and} \quad d\bar{z}_j(\theta_k) = 0 \]

\[ d\bar{z}_j(\theta_k) = 0 \quad \text{and} \quad dz_j(\theta_k) = \delta_{jk} \]

\[ eg, \quad dz_j \left( \frac{1}{2}(\partial x_j - i \partial y_j) \right) = \frac{1}{2} + i \left( -\bar{z}_j \right) = 1 \]

Similarly, \(\Lambda^k T^* M \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^p (T^* M)^{1,0} \otimes \Lambda^q (T^* M)^{0,1}\)

A form of type \((p,q)\) in local holomorphic coordinates

\[ \sum \alpha_{j_1, \ldots, j_p, k_1, \ldots, k_q} dz_{j_1} \wedge \cdots \wedge dz_{j_p} \wedge d\bar{z}_{k_1} \wedge \cdots \wedge d\bar{z}_{k_q} \]

\[ = \sum \alpha_{\alpha, \beta} dz_{\alpha} \wedge d\bar{z}_{\beta}, \quad \alpha, \beta \in \mathbb{N}_0^m, \quad m = \dim M \]
If $M$ is a complex manifold then $T^{1,0}M$ is a holomorphic bundle, and so are $\otimes^k T^{1,0}M$ and $\Lambda^k T^{1,0}M$.

$\text{O}(T^*_0H)$ is just a smooth bundle.

The composition of anti-holomorphic maps is not anti-holomorphic, so there is no notion of "anti-holo" bundle.

We write $\Omega^{1,1}_k(M) = C^\infty(M; \Lambda^{k,0}T^*_0M)$

On an arbitrary manifold with a complex structure $J$, the exterior derivative has four type components:

$$d : \Omega^{1,1}_k(M) \to \Omega^{1,2}_k(M) \oplus \Omega^{0,1}_k(M) \oplus \Omega^{1,0}_k(M) \oplus \Omega^{2,0}_k(M)$$

E.g., let's consider a 1-form $\omega \in \Omega^1(M)$

$\omega$ has type $(1,0)$ if it vanishes on $JT^0M$, of type $(0,1)$

i.e., $\omega(V) = \omega(\pi_{1,0}V)$, $(\pi_{1,0}\omega)(V) = \omega(\pi_{0,1}V)$

If $\omega$ has type $(1,0)$, what about $d\omega$?

$$d\omega(V_1, V_2) = L_{\bar{V}_2}(\omega(V_1)) - L_{\bar{V}_1}(\omega(V_2)) - \omega([V_1, V_2])$$

$$\pi_{2,0}d\omega(V_1, V_2) = d\omega(\pi_{1,0}V_1, \pi_{0,1}V_2)$$

$$\pi_{1,1}d\omega(V_1, V_2) = d\omega(\pi_{1,0}V_1, \pi_{0,1}V_2) + d\omega(\pi_{0,1}V_1, \pi_{1,0}V_2)$$

None of these can be guaranteed to vanish.

If $\omega$ has type $(1,0)$ then $\pi_{0,1}d\omega(V_1, V_2) = -\omega(\pi_{0,1}V_1, \pi_{0,1}V_2)$

so if $T^{0,1}M$ is involutive, $i.e.$, $J$ is integrable, then $\pi_{0,1}d\omega = 0$.

$\text{O}(T^*_0H)$, if $\pi_{0,1}d\omega = 0$ for all $\omega$ of type $(1,0)$ then

$$\pi_{1,0}[\pi_{0,1}V_1, \pi_{0,1}V_2] = 0 \iff \forall V_1, V_2, i.e., T^{0,1}M \text{ is involutive.}$$
Then let \((M, J)\) be a mfd with almost C-str. TFAE:

1) \(M\) has a complex str inducng \(J\)
2) \(T^0 M\) is involutive
3) \(T^m M\) is involutive
4) \(J : \mathcal{L}^1 (M) \to \mathcal{L}^{-1} (M) \otimes \mathcal{L}^1 (M)\)
5) \(\mathcal{L}^2 (M) \to \mathcal{L}^0 (M) \otimes \mathcal{L}^2 (M)\)
6) \(\mathcal{L}^{0,2} (M) \to \mathcal{L}^{-1,1} (M) \otimes \mathcal{L}^{0,1} (M)\)

For any \(p, q\)

In this case we write \(d = \partial + \bar{\partial}\)

\(\bar{\partial} : \mathcal{L}^0 (M) \to \mathcal{L}^{-0,1} (M)\)

(by def, \(\partial \bar{\omega} = \bar{\partial} \omega\))

These satisfy their own Leibniz rule: if \(\omega, \eta \in \mathcal{L}^p (M)\)

\(\partial (\omega \eta) = \partial \omega \eta + (-1)^p \omega \partial \eta\)

\(\bar{\partial} (\omega \eta) = \bar{\partial} \omega \eta + (-1)^p \omega \partial \eta\)

Moreover, since \(d^2 = 0\) we have \((\partial + \bar{\partial})^2 = 0 = \partial^2 + \bar{\partial}^2 + 2 \partial \bar{\partial}\)

At computing type we see that \(\partial^2 = 0, \bar{\partial} \partial + \bar{\partial} ^2 = 0, \bar{\partial}^2 = 0\)

The Dolbeault complex of holomorphic p-forms is

\[0 \to \mathcal{L}^0 (M) \xrightarrow{\partial} \mathcal{L}^1 (M) \xrightarrow{\bar{\partial}} \mathcal{L}^2 (M) \xrightarrow{\partial} \mathcal{L}^3 (M) \to 0\]

The Dolbeault cohomology groups are

\[H^{p,q}_\mathbb{C} (M) = \frac{\bar{\partial} \text{- closed \((p,q)\)-forms}}{\bar{\partial} \text{- exact \((p,q)\)-forms}} = \frac{\ker (\bar{\partial} : \mathcal{L}^p (M) \to \mathcal{L}^{p,q+1} (M))}{\text{Image} (\bar{\partial} : \mathcal{L}^{0,q} (M) \to \mathcal{L}^{0,q+1} (M))}\]

In particular

\[H^{0,0}_\mathbb{C} (M) = \text{holomorphic sections of } \mathcal{N}^0 (T^0 M)^*\]

We will show that if \(M\) is a closed complex manifold

then \(H^{p,q}_\mathbb{C} (M)\) is a finite-dim C-v.s.
N.B. There is no natural map between $H^*_\text{deRham}(M)$ and $H^*_\partial(M)$ on general complex manifolds.

We can relate them using other cohomology theories:
From $\bar{\partial} \circ \partial = 0$ we note that
1) $\partial \bar{\partial}(\partial + \bar{\partial}) = 2\bar{\partial}^2 = 0$ so we define Bott-Chern cohomology
$$H^p_{BC}(M) = \frac{\ker (\partial + \bar{\partial} : \Omega^{p, q}(M) \to \Omega^{p+1, q}(M) \oplus \Omega^{p, q+1}(M))}{\text{Im} (\partial + \bar{\partial})}$$

2) $(\partial + \bar{\partial}) \bar{\partial} = 0$ so we define Aeppli cohomology
$$H^p_{\text{Aeppli}}(M) = \frac{\ker (\bar{\partial} : \Omega^{p, q}(M) \to \Omega^{p, q+1}(M))}{\text{Im} (\partial + \bar{\partial}) \cap \Omega^{p, q}(M)}$$

There are natural maps \( \oplus \):
\[
\begin{align*}
\oplus H^p_{BC}(M) & \quad \oplus H^p_{\text{deRham}}(M) \quad \oplus H^p_{\partial}(M) \\
\oplus H^p_{\text{Aeppli}}(M) \quad \oplus H^p_{\text{Aeppli}}(M) \quad \oplus H^p_{\text{Aeppli}}(M) \\
\oplus H^p_{\text{Aeppli}}(M) \quad \oplus H^p_{\text{Aeppli}}(M) \quad \oplus H^p_{\text{Aeppli}}(M)
\end{align*}
\]

Using cohomological algebra (a close store) you can check [DGMS] that if \( \oplus \) is injective then all maps in the diagram are isomorphisms.
The map $\otimes$ is injective if $\ker \partial \cap \ker \bar{\partial} \cap \text{Im } d = \text{Im } \partial \bar{\partial}$. We say that a manifold satisfies the $\partial \bar{\partial}$-lemma if this is true. Hence, if a manifold satisfies the $\partial \bar{\partial}$-lemma,

then $H^k_{d\bar{d}}(M) = \bigoplus_{p+q=k} H^p_{\bar{\partial}}(M)$.