Last Time: Almost complex structures
If $M$ is a smooth mfld, an almost complex structure is a
bundle map $J: TM \to TM$ s.t. $J^2 = -\text{Id}$
If $M$ is a complex mfld then the underlying smooth mfld
inherits an almost complex structure which is called integrable
If the real dimension of $M$ is two then every a.C-str is integrable
This Time: Newlander-Nirenberg Thm for analytic manifolds
Given $J$ we define
$T^\perp J M = \{ w \in T_0 M : J(w) = -w \}$
N-N then says that $J$ is integrable $\iff T^\perp J M$ is involutive
(i.e., if $V, W$ are sections of $TM$ then so is $[V, W]$)

Let's start by recalling a Thm from differential geometry
Suppose $M$ is a smooth mfld & $E \subset TM$ is a subbundle (of rank $k$)
We say that
1) $E$ is involutive if the Lie bracket of two sections of $E$ is a section of $E$
2) $E$ is integrable if each pt of $M$ has a nbhd $U$ & a map $\Phi_u : U \to \mathbb{R}^{2k}$
s.t. $E|_U = \text{Ker } D\Phi_u$
(i.e., each fiber $\Phi_u^{-1}(v)$ is a submfld of $U$ & tangent space $E|_{\Phi_u^{-1}(v)}$)
The Frobenius theorem says that $E$ is involutive $\iff E$ is integrable
Given the Frobenius theorem for smooth manifolds, we can deduce a Frobenius theorem for complex manifolds.

Let $\mathcal{M}$ be a complex n-dimensional C-dim $n$, $E \in T^{\text{hol}} \mathcal{M}$ a subbundle of $\mathcal{C}$-rank $k$. Then $E$ is involutive if $\bar{F}$ is holomorphically integrable (i.e., we have local holomorphic maps $\phi_x: \mathcal{U} \to \mathcal{C}^{n\times k}$ s.t. $E_x = \text{Ker}(D\phi_x)$). Then $\phi$ is holomorphic.

Theorem: Let $\text{Re} : T^{\text{hol}} \mathcal{M} = T^{\text{hol}} \mathcal{M} \to \mathcal{M}$

$\omega \mapsto \text{Re}(\omega)$

& if $E$ is involutive then so is $\text{Re} E$

Next we want to put a complex structure on $\text{Image}(\phi_x) = V \subseteq \mathbb{C}^{n\times k}$

for which $\phi_x$ is holomorphic.

We can identify $T\phi_x(\mathcal{U}) V = T\mathcal{U} / \text{Re} E_x$

Since $E$ is a holomorphic subbundle, the integrable $\mathcal{C}$-str on $\mathcal{M}$ preserves $\text{Re} E$ hence $\mathcal{J}$ descends to $\mathcal{V}$

Thus $T\phi_x(\mathcal{U}) V$ inherits a complex structure (by the point 3).

Let by construction $D\phi_x$ commutes with $\mathcal{J}$

so $\phi_x$ is holomorphic.

We will say that $(\mathcal{M}, \mathcal{J})$ is real analytic if $\mathcal{M}$ has an atlas, whose transition maps are real analytic

& in each of these coordinate charts, $\mathcal{J}$ is a real analytic family of matrices.
Then if $J$ is an almost complex structure on $M \in C.M$ is $R$-analytic
Then $J$ is integrable iff $T^\alpha M$ (or $T^\beta M$) is involutive.

If (Weit, see Voisin)
It’s enough to work locally, so assume $M = U \subseteq \mathbb{R}^{2n}$ open. $0 \in U$
$\tilde{J}$ is a real analytic matrix-valued map satisfying $\tilde{J}^2 = -\text{Id}$ given
by a convergent power series.

Hence there’s a neighborhood $\tilde{U} \subseteq \mathbb{C}^{2n}$ on which
this power series converges, call the extension $\tilde{J}$

Let $\tilde{E}$ be the $\mathbb{C}$-ideal of $\tilde{J}$

so $E = \tilde{E} \cap \mathbb{C}^{2n}$

Sections of $\tilde{E}$ over $\tilde{U}$ are $v$ of $\mathbb{R}^n$ form $V + i\tilde{J}(V)$

where $V$ is a $C$-v.f. over $\tilde{U}$

Hence the involutivity of $T^\alpha U$ implies the involutivity of $\tilde{E}$

Thus, up to shrinking the nbhd, we know that there exists

a holomorphic function $\Phi: \tilde{U} \rightarrow \mathbb{C}^n$

whose fibers are the integral subvarieties of $\tilde{E}$, s.t. $\tilde{E} = \ker(D\Phi)$

Now note that $U$ sits in $\tilde{U}$ like $(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ in $(\mathbb{C}^{2n}, \mathbb{C}^{2n})$

$\tilde{U}$ is transverse to $\tilde{E}$ (i.e. $U + \tilde{E}$)

hence the restriction $\Phi | U = \Phi: U \rightarrow \mathbb{C}^n$ is a diffeomorphism.

Finally note that the derivative of $\Phi$

$D\Phi: T_\tilde{U} \rightarrow T_{\mathbb{C}^n}$ identifies $\tilde{J}$ the complex str on $\mathbb{C}^n$

This, follows from $\mathbb{C}$-linearity of $T_\tilde{U} \rightarrow T_\tilde{U}/\mathbb{R}E_2$

$\tilde{U}$ in the quotient $T_\tilde{U}/\mathbb{R}E_2$

we have $V = -i\tilde{J}(V)$ i.e. $iV = \tilde{J}(V)$