Let's start by noticing that a complex vector space is the same as a real vector space together with a "complex structure" in the form of an endomorphism $J$ s.t. $J^2 = -\text{Id}$.

Indeed, given $J$ we can define $(a+ib) \cdot v = av + bJ(v)$

$J$ thus makes the $\mathbb{R}$-v.s. a $\mathbb{C}$-v.s.

If we start with a $\mathbb{C}$-v.s. then multiplication by $i$ is an $\mathbb{R}$-lin. endomorphism of the underlying $\mathbb{R}$-v.s. that squares to $-\text{Id}$.

If we have $V$ a $\mathbb{R}$-v.s. of dim $2n$ and a $C$-str $J$

$V \otimes \mathbb{C}$ is a $\mathbb{C}$-v.s. of dim $2n$

$J$ it has two complex structures.

$J$ extends to $V \otimes \mathbb{C}$ by $J(v \otimes \alpha) = J(v) \otimes \alpha$

$i$ acts from the $\mathbb{C}$-factor acts on $V \otimes \mathbb{C}$ by $i(v \otimes \alpha) = v \otimes i\alpha$
$J^2 = -I \Rightarrow J$ is diagonalizable with two eigenvalues $i$ and $-i$.

Denote the eigenspaces by $V \otimes \mathbb{C} = V^{i} \oplus V^{-i}$

$V^{i} = \{ w \in V \otimes \mathbb{C} : J(w) = i \, w \}$

$V^{-i} = \{ w \in V \otimes \mathbb{C} : J(w) = -i \, w \}$

Thus $V^{i}$ is the subspace where the two complex structures coincide & $V^{-i}$ the subspace where they don't.

Notice that we can define conjugation on $V \otimes \mathbb{C}$

by $\overline{v \otimes \mathbf{a}} = \overline{v} \otimes \overline{\mathbf{a}}$ & then $V^{i} = V^{-i}$

so $\dim_{\mathbb{C}} V^{i} = \dim_{\mathbb{C}} V^{-i} = \frac{1}{2} \dim_{\mathbb{C}} V \otimes \mathbb{C} = n$

We have an iso of $\mathbb{C}$-v.s.: $(V, J) \xrightarrow{\cong} (V^{i}, i)$

$v \mapsto \frac{1}{2} (v - iJv)$

E.g., if we start with $\mathbb{C}^n = \{ (w_1, \ldots, w_n) : w_j \in \mathbb{C} \}$

& decompose $w_j = a_j + i b_j \wedge a_j, b_j \in \mathbb{R}$ then we have the actual identification with $\mathbb{R}^{2n} = \{ (a_1, b_1, \ldots, a_n, b_n) : a_j, b_j \in \mathbb{R} \}$

Scalar multiplication by $i$ in $\mathbb{C}^n$ induces the complex structure $J : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$, $J(a_1, b_1, \ldots, a_n, b_n) = (-b_1, a_1, \ldots, -b_n, a_n)$

Now consider the complexification $\mathbb{R}^{2n} \otimes \mathbb{C} = \mathbb{C}^{2n}$

The standard basis of $\mathbb{R}^{2n}$: $x_1, y_1, \ldots, x_n, y_n$

induces the standard basis of $\mathbb{C}^{2n}$: $x_1 = x_1 \otimes 1, y_1 = y_1 \otimes 1, \ldots, y_n = y_n \otimes 1$

$I$ extends to $\mathbb{C}^{2n}$: $J(x_k) = y_k, J(y_k) = -x_k$

hence its $i$-eigenspace is span $\{ x_k - i y_k = (x_k \otimes 1 - y_k \otimes i) \}$

& its $-i$-eigenspace is span $\{ x_k + i y_k = (x_k \otimes 1 + y_k \otimes i) \}$

and we note $x_k - i y_k = x_k + i y_k$ so these are conjugate.
Every R-vs of even dimension can be given a complex structure.

**Def** An almost complex structure on a smooth manifold $M$ is a vector bundle isomorphism $J: TM \rightarrow TM$ s.t. $J^2 = -\text{Id}$

This turns $TM$ into a C-vector bundle, but does not turn $M$ into a C-mfld.

Clearly if $M$ admits an almost C-str it must be even dim. orientable but this is not enough. 

E.g. the only spheres that admit an almost complex structure are $S^2$ & $S^6$ (Borel-Serre, might prove later)

A map between smooth manifolds $M$ with almost complex structures $F: (M, J) \rightarrow (M', J')$ is called almost complex (pseudoholomorphic) if $DF \circ J = J' \circ DF$

**Prop** If $U \subseteq \mathbb{C}^n$, $V \subseteq \mathbb{C}^n$ are open sets, a map $F: U \rightarrow V$ is pseudoholomorphic iff it is holomorphic

In particular a complex manifold $M$ induces an almost complex structure on its underlying smooth manifold

is a complex structure $\Rightarrow$ almost complex structure

We say that an almost complex structure is integrable iff it comes from a complex structure
Rmk: If $\mathbf{M}$ is a $C$-man, then $T^{\mathrm{hol}}\mathbf{M} = T^{1,0}\mathbf{M} = (\mathcal{T}M \otimes \mathbb{C})^*$.

Not every almost complex structure is integrable, but there is a nice characterization.
Recall that $\nu$'s have a bracket:

$\nu, \omega \in C^\infty(\mathbf{M}; \mathcal{T}\mathbf{M})$, $[\nu, \omega]$ is the $\nu$ satisfying

$\mathcal{L}_\nu \omega = \nu \mathcal{L}_\nu \omega$.

This easily extends to sections of $\mathcal{T}M \otimes \mathbb{C}$.

The Newlander–Nirenberg theorem says that an almost complex structure is integrable if the bracket of two sections of $T^{1,0}\mathbf{M}$ is another section of $T^{1,0}\mathbf{M}$.

Let's think about what it means to be integrable on $\mathbb{R}^2$.

A complex structure on $\mathbb{R}^2$ is a way to decide which functions are holomorphic.

$f$ is holo $\iff \nabla f = (\partial_x + i \partial_y) f = 0$.

How can we recognize this in other coordinates?

That is, suppose we are given two real vector fields:

$Q_j = a_j(x, y) \partial_x + b_j(x, y) \partial_y, \quad j \in \{1, 2\}$

Let $Pf = (Q_1, Q_2, Q_3)f$.

Can we find coordinates $\mathbf{x} = (x, y), \mathbf{v} = (x, y)$ so that in these coordinates, $Pf = 0$ is equivalent to $(\partial_x + i \partial_y) f = 0$?

A necessary condition is that $Q_1, Q_2$ be lin. ind.

It turns out that this is also sufficient.

(In this case $T^{1,0}\mathbf{M}$ is always closed under Lie bracket.)
Suppose we can solve \( Pw = 0 \) if \( w = u + iv \), \( u, v \) \( \mathbb{R} \)-valued
and \( \nabla u, \nabla v \) lin. ind.

Then we'll use \( u \) & \( v \) as the new coordinates.

On the one hand, by the chain rule
\[
P = \alpha(u,v) \partial_u + \beta(u,v) \partial_v \text{ for some } \alpha, \beta
\]
\[
(\partial_x \rightarrow \frac{\partial}{\partial x} \partial_u + \frac{\partial}{\partial y} \partial_v, \partial_y \rightarrow \frac{\partial}{\partial y} \partial_u + \frac{\partial}{\partial y} \partial_v)
\]

On the other hand, since \( Pw = 0 \Rightarrow \beta(u,v) = 0 \),
\[
\text{then } P = -i\beta(\partial_u + i\partial_v)
\]

And so \( Pf = 0 \Rightarrow (\partial_u + i\partial_v)f = 0 \) (\( \beta \neq 0 \) by lin. ind.)

The only gap is that we need to be able to solve (locally)
\( Pw = 0 \).

Since \( \alpha, \beta \) are lin. ind., the solvability together with
\( \Delta u, \Delta v \) lin. ind. follows from local solvability of
elliptic equations (later).