Another way of thinking about manifolds, real or complex, is line with modern algebraic geometry, is through “geometric structure.”

**Def:** \( F \in \mathbb{R}, \mathbb{C} \), \( X \) a topological space.

For any \( U \subseteq X \) open, let \( C(U) = C^0(U) \) denote the continuous functions \( U \rightarrow \mathbb{R} \).

A geometric structure \( \mathcal{C} \) on \( X \) is a collection of subrings

\[ \mathcal{C}(U) \subseteq C(U) \quad \forall \ U \subseteq X \text{ open}, \]

satisfying:

1. The constant function are in \( \mathcal{C}(U) \).
2. If \( f \in \mathcal{C}(U) \) & \( V \subseteq U \) open, then \( f |_V \in \mathcal{C}(V) \).
3. If \( \{U_i\} \) is a collection of open subsets of \( X \), \( U = \bigcup_i U_i \).

\( \forall \) we are given \( f_i \in \mathcal{C}(U_i) \) s.t. \( f_i |_{U_j} = f_j |_{U_j} \) whenever \( U_i \cap U_j \neq \emptyset \).

Then \( \exists ! \) \( f \in \mathcal{C}(U) \) s.t. \( f |_{U_i} = f_i \forall i \).

The pair \( (X, \mathcal{C}) \) is called a geometric space.

A functions in \( \mathcal{C}(U) \) will sometimes be called distinguished.

(iii) \( \mathcal{C}(iii) \) tell us that being distinguished is an open property.

Typical examples are differentiability & analyticity.

In the language of sheaves (which we will discuss later),

\( \mathcal{C} \) is a subsheaf of the sheaf of continuous functions.
Ex: 1) If \( U \subseteq \mathbb{R}^n \) open \& let \( C^\infty \) be the geometric structure \( \nu \mapsto C^\infty(\nu) \) then \( (U, C^\infty) \) is a geometric space.

2) If \( U \subseteq C^m \) open \& let \( \Theta \) be the geometric structure \( \nu \mapsto \Theta(\nu) = \text{holomorphic function} \) \( \nu \mapsto \Theta(\nu) \) is a geometric space.

A morphism of geo spaces \( f : (X, C^x) \rightarrow (Y, C^y) \)

is a continuous map \( f : X \rightarrow Y \) with the property that

if \( g \in C^y(W) \) then \( g \circ f \in C^x(f^{-1}(W)) \)

(we write this map as \( f^* : C^y(U) \rightarrow C^x(f^{-1}(W)) \))

\( f \) is an isomorphism if there is an inverse morphism.

Ex: 1) \( f : (U, C^\infty_U) \rightarrow (V, C^\infty_V) \) is the same as \( f \in C^\infty(U, V) \)

2) \( f : (U, \Theta_U) \rightarrow (V, \Theta_V) \) \( f : U \rightarrow V \) holomorphic

3) If \( U \subseteq X \) open \( \Rightarrow (U, C^x(U)) \rightarrow (X, C^x) \) is a morphism

4) If \( X \) is a real (complex) manifold \( \Rightarrow \exists \{ U_a : U_a \rightarrow V_a \} \)

\( U_a \subseteq X, V_a \subseteq F^* \)

We define a geometric structure \( C^\infty_x \) (\( C^x \) resp.) as follow:

For \( U \subseteq X \) open, define

\[
C^\infty_x(U) = \{ f \in C^\infty(U) : (f|_{UM_a}) \circ \varphi^{-1}_a : U_a(UM_a) \rightarrow \mathbb{R} \text{ smooth \& } UM_a \neq \emptyset \}
\]

\( \Theta_x(U) = \{ \text{holo.} \} \)

**Equiv Def**: A smooth (complex) nfd of \( F \)-dim \( n \) is a co str. \((X, C^x)\)
in which every pt \( x \in X \) has a nbd \( U \) s.t. \((U, C^\infty_U) \cong (\mathbb{R}^n, C^\infty) \), \( U \subseteq \mathbb{R}^n \) open

or s.t. \((U, \Theta_U) \cong (\mathbb{R}^n, \Theta_U), U \subseteq C^m \) open.
Given a geo space \((X, C_\mathbb{A})\) & a point \(z \in X\) we can localize \(C_\mathbb{A}\) to \(z\):

We define \(C_\mathbb{A}_z\) as a ring of equivalence classes of functions \(f\). Each \([f] = [f]_z \in C_\mathbb{A}_z\) is represented by \(f \in C_\mathbb{A}(U), z \in U\) open & two representatives \(f_1, f_2 \in C_\mathbb{A}(U)\) are equivalent if there is an open \(W, z \in W, \) s.t. \(f_1|_W = f_2|_W\)

We call \(C_\mathbb{A}_z\) the stalk of \(C_\mathbb{A}\) at \(z\), \([f]_z\) the germ of \(f\) at \(z\).

A derivation of \(C_\mathbb{A}_z\) is an \(\mathbb{F}\)-linear map \(D : C_\mathbb{A}_z \rightarrow \mathbb{F}\) that satisfies the Leibnitz rule at \(z\):

\[D([[f][g]]) = f(z)D([g]) + D([f])g(z) \quad \forall [f], [g] \in C_\mathbb{A}_z\]

The real tangent space of \(X\) at \(z\) is the \(\mathbb{R}\)-val. of derivs. of \(C_\mathbb{A}_z\)

\[
\begin{align*}
\text{Ex: 1) For } U \subseteq \mathbb{R}^n \text{ open, } D_j : C_\mathbb{R} \rightarrow \mathbb{F}, \quad D_j([[f]]) = \frac{\partial f}{\partial x_j}(z) \quad \text{is a derivation} \\
2) \text{For } U \subseteq C^n \text{ open, } D_j : \mathcal{O}_X \rightarrow \mathbb{F}, \quad D_j([[f]]) = \frac{\partial f}{\partial z_j}(z) \quad \text{is a derivation}
\end{align*}
\]

NB: For a manifold, the stalk of \(C_\mathbb{A}\) is the same as the stalk of \(C_\mathbb{A}(U)\), a coord chart containing \(z\).

Let \(J_z \subseteq C_\mathbb{A}_z\) be the ideal of germs that vanish at \(z\)

\[J_z = \{ [f] \in C_\mathbb{A}_z : [f] = 0 \} \text{ for some } [g], [h] \in J_z \] defined by

The map \(C_\mathbb{A}_z \rightarrow C_\mathbb{A}_z/J_z\) is evaluation at \(z\) \n
\([f] \mapsto f(z)\)
The map \( J_2 \rightarrow J_\mathbb{R} / J_\mathbb{R} \), for a manifold is the "total derivative" (exterior derivative).

Given \( f \in \mathcal{C}^0(U) \), \( U \subseteq \mathbb{F}^n \), we can write
\[
f(\omega) = f(\xi) + (\partial_{\omega_j} f)(\xi)(\omega_j - \xi_j) + \ldots + (\partial_{\omega_r} f)(\xi)(\omega_r - \xi_r) + \text{higher order terms}
\]

If \( \xi \in J_\mathbb{R} \) then \( f(\xi) = 0 \) while the remainder \( \eta \in J_\mathbb{R}^2 \), so the class of \( [\eta] \) in \( J_\mathbb{R}^2 / J_\mathbb{R} \).

1) The class of \( [(\partial_{\omega_j} f)(\xi)(\omega_j - \xi_j) + \ldots + (\partial_{\omega_r} f)(\xi)(\omega_r - \xi_r)] \)
\[
= \sum_{j=1}^n (\omega_j - \xi_j) \frac{\partial f}{\partial \omega_j}(\xi) d\omega_j
\]

Thus \( J_\mathbb{R}^2 / J_\mathbb{R}^2 \cong \mathbb{F}^n \) with each choice of coordinates inducing a basis of \( J_\mathbb{R}^2 / J_\mathbb{R}^2 \).

Claim: \( \text{Der}(\mathcal{A}_\mathbb{R}) \cong (J_\mathbb{R} / J_\mathbb{R}^2)^* = \text{Ann.} \) of \( J_\mathbb{R}^2 \) in \( J_\mathbb{R}^* \)

Proof: \( \text{Der}(\mathcal{A}_\mathbb{R}) \xrightarrow{\text{res}} J_\mathbb{R}^* \) restriction, by the Leibniz rule.

Element in the image vanish on \( J_\mathbb{R}^2 \).

Conversely, given \( \phi \in J_\mathbb{R}^* \) s.t. \( \phi|_{J_\mathbb{R}^2} = 0 \)

Define \( D: \mathcal{A}_\mathbb{R} \rightarrow \mathbb{F} \) by \( D([f]) = \phi([f] - [f(\xi)]) \)

Note \( D([f][g]) = \phi([f] - [f(\xi)]) \phi([g] - [g(\xi)]) \)
\[
= \phi(f(\xi) \phi([g] - [g(\xi)]) + f(\xi) g(\xi)) \phi([f] - [f(\xi)])
\]

Thus \( J_\mathbb{R} / J_\mathbb{R}^2 \cong \mathbb{F}^n \) with each choice of coordinates inducing a basis of \( J_\mathbb{R} / J_\mathbb{R}^2 \).