We mentioned last time

**Theorem (Hartogs)**

If \( U \subseteq \mathbb{C}^n \) is open, \( n \geq 2 \), \( K \subseteq U \), \( U \setminus K \) connected

then any holomorphic function on \( U \setminus K \) has a unique holomorphic extension to \( U \)

**Example.** Consider \( H = \{ (z, \omega) \in \mathbb{C}^2 : |z| < 1, \frac{1}{2} < |\omega| < 1 \} \cup \{ z(1, \omega) : 0 < |\omega| < 1 \} \)

Let \( f \) be holomorphic on \( H \)

Claim: \( \exists F \) holomorphic on \( D = \mathbb{D}(1, 1) \)

s.t. \( F|_H = f \)

In fact \( F(z, \omega) = \frac{1}{2\pi i} \int_{|\zeta| = \frac{1}{2}} \frac{f(z, \zeta)}{\zeta - \omega} \, d\zeta \quad \forall \quad (z, \omega) \in (\frac{1}{2}, 1) \)

\( F \) is holomorphic. In deed, \( \partial_{\bar{\omega}} \left( \frac{f(z, \zeta)}{\zeta - \omega} \right) = \partial_{\omega} \left( \frac{f(z, \zeta)}{\zeta - \omega} \right) = 0 \)

for any fixed \( z \) with \( 0 < |z| < \frac{1}{2} \)

\( \omega \rightarrow f(z, \omega) \) is holomorphic on all of \( \{ |\omega| < 1 \} \)

so \( F(z, \omega) = f(z, \omega) \) for any \( 0 < |z| < \frac{1}{2}, |\omega| < 1 \)

by the Cauchy int. formula. Hence \( F = f \) on all \( H \).

**Added later:** As Irving Yang pointed out in class, this doesn't quite fit into Hartogs' extension theorem. To guarantee a holomorphic extension into \( D \) the theorem asks us to start with a function holomorphic in a neighborhood of \( \partial(D) \); here we're getting away by asking for something else.
Lemma: Let $f \in C^1(\mathbb{R})$ then
\[
\int_{\mathbb{R}} f(x) \, \mathrm{d}x = \int_{\mathbb{R}^2} \frac{\partial f}{\partial y} \, \mathrm{d}x \, \mathrm{d}y = \frac{2\pi}{\mathcal{L}} \int_{\mathbb{R}} f(x) \, \mathrm{d}x.
\]

Prop: Let $u \in C^1(\mathbb{R})$ then
\[
u(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{u(y)}{y-z} \, \mathrm{d}y + \frac{1}{\pi} \int_{\mathbb{R}} \frac{\partial u}{\partial y}(z) \, \mathrm{d}x \, \mathrm{d}y.
\]

Proof: Fix $z$, let $\epsilon < \mathcal{L}(z, \partial \mathcal{C})$ & let $\Delta_{\epsilon} = \{ z \in \mathcal{C} : |z-z_0| < \epsilon \}

Apply Lemma to $f(z) = \frac{u(z)}{z-\Omega}$
\[
\int_{\Delta_{\epsilon}} \frac{u(z)}{z-\Omega} \, \mathrm{d}z = \frac{2\pi}{\mathcal{L}} \int_{\Delta_{\epsilon}} \frac{\partial u}{\partial y}(z) \, \mathrm{d}x \, \mathrm{d}y.
\]

As $\epsilon \to 0$ the LHS converges to $\int_{\mathbb{R}} \frac{u(z)}{z-\Omega} \, \mathrm{d}z + 2\pi i \, u(z) * f(z).

Thus let $\phi \in C^\infty_c(\mathbb{C})$, let $u(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\phi(z)}{z-\Omega} \, \mathrm{d}x \, \mathrm{d}y$

then $u$ is an analytic function outside of supp $\phi$

& $u$ is smooth on $\mathbb{C}$ & $\partial \nabla u = \phi$

Proof: Interchanging derivatives & the integral we see that
$u \in C^\infty_c(\mathbb{C})$ & by a change of variables we have
\[
u(z) = -\frac{1}{\pi} \int_{\mathbb{R}} \phi(\xi-z) \, \mathrm{d}x \, \mathrm{d}y.
\]

so $\partial \nabla^2 u(z) = -\frac{1}{\pi} \int_{\mathbb{R}} \partial \nabla \phi(\xi-z) \, \mathrm{d}x \, \mathrm{d}y = \frac{1}{\pi} \int_{\mathbb{R}} \partial \nabla \phi(z) \, \mathrm{d}x \, \mathrm{d}y$

apply the proposition to any disc containing supp $\phi$

& we get that this equals $\phi(z)$

i.e. $\partial \nabla u = \phi$.\]
N.B.: Even though \( \phi \) has compact support

No solution of \( \partial_{z_j} u = \phi \) can have compact support if \( \int \phi \neq 0 \)

Indeed, if \( u(z) = 0 \) \( \forall |z| > R \) then

\[
0 = \int_{|z| < R} u(z) \, dz = \int_{\partial B_R(0)} \partial_{z_j} u \, dS = 2\pi R \int \phi \, dS
\]

Then suppose \( f_j \in C^\infty_c (\mathbb{C}^n), \ j \in \{1, \ldots, n\}, \ n > 1 \)

s.t. \( \partial_{z_j} f_k = \partial_{\bar{z}_k} f_j \ \forall \ j, k \in \{1, \ldots, n\} \)

then there is a \( u \in C^\infty_c (\mathbb{C}^n) \) s.t. \( \partial_{z_j} u = f_j \ \forall j \in \{1, \ldots, n\} \)

Proof:

Define \( u(z) = \frac{1}{2\pi i} \int_{|z| = R} \frac{f_j(z_1, z_2, \ldots, z_n)}{z - z_1} \, d\bar{z}_j \)

\[
= \frac{1}{2\pi i} \int_{|z| = R} \frac{f_j(z - z_1, z_2, \ldots, z_n)}{z} \, d\bar{z}_j
\]

We note that \( u \in C^\infty_c (\mathbb{C}^n) \)

It since \( f_j \) has compact support, \( u \) vanishes if \( |z_1 + \ldots + z_n| \gg 0 \)

By the previous then, \( \partial_{z_j} u = f_j \)

Also differentiating, \( \partial_{\bar{z}_j} u = \frac{1}{2\pi i} \int_{|z| = R} \partial_{\bar{z}_j} f_j(z_1, \ldots, z_n) \, d\bar{z}_j \)

Hence \( u \) solves the system of equations

Let \( \mathcal{K} = 0 \) supp \( f_j \)

\( u \) is holomorphic on \( \mathbb{C}^n \setminus \mathcal{K} \)

We know that \( u \) is zero if \( |z_1 + \ldots + z_n| \gg 0 \)

So by the identity theorem \( u \) must vanish on the unbounded component of \( \mathbb{C}^n \setminus \mathcal{K} \Rightarrow u \in C^\infty_c (\mathbb{C}^n) \)