Let \( f_i(x_1, \ldots, x_n) \) be polynomials with coefficient in \( \mathbb{R} \) or \( \mathbb{C} \). An affine algebraic variety is the common zero set of \( \mathbf{X} = \mathbf{X}(f_1, \ldots, f_k) = \{ x : f_i(x) = 0 \ \forall i \} \).

We incorporate the field into the notation:
- \( \mathbf{X}(\mathbb{C}) = \{ x \in \mathbb{C}^n : f_i(x) = 0 \ \forall i \} \)
- \( \mathbf{X}(\mathbb{R}) = \{ x \in \mathbb{R}^n : f_i(x) = 0 \ \forall i \} \) (when \( f_i \in \mathbb{R}[x] \))

Two "incarnations" of the variety.

These can be thought of naturally as topological spaces with the topology they inherit from \( \mathbb{C}^n \) or \( \mathbb{R}^n \).

(Alternately, you can use the Zariski topology, induced by declaring that zero sets of polynomials are closed.)

It is frequently inconvenient that \( \mathbf{X}(\mathbb{C}) \) is essentially never compact.

To remedy this, we shift our attention to projective space:

\[
\mathbb{P}^n = \mathbb{C}^{n+1} \setminus \{ 0 \mathbf{s} \} = \{ (z_0, \ldots, z_n) \in \mathbb{C}^{n+1} : \mathbf{s} \neq 0 \}
\]

\[
\mathbb{P}^n = \mathbb{C}^{n+1} / \mathbb{C}^* \text{, hence compact}
\]

Given homogeneous polynomials \( F_i(x_0, \ldots, x_n) \)

we obtain a "projective variety"

\[ \mathbf{X} = \mathbf{X}(F_1, \ldots, F_k) = \{ x \in \mathbb{C}P^n : F_i(x) = 0 \ \forall i \} \]

As before, if the polynomials have real coefficients,

\[ \mathbf{X}(\mathbb{R}) = \{ x \in \mathbb{R}P^n : F_i(x) = 0 \ \forall i \} \].
We can ask about the relation between the topology and geometry of $X(\mathbb{R}), X(\mathbb{C})$ and the algebraic properties of $X$.

For example, say $X$ is the zero set of a single homogeneous polynomial of degree $d$, $F$. Can we recover $d$ from $X(\mathbb{C}), X(\mathbb{R})$? (Only a sensible question for $F$ irreducible.)

It turns out that $X_F(\mathbb{C})$ determines a homology class $[X_F(\mathbb{C})] \in H_{2n-2}(\mathbb{C}P^n; \mathbb{Z})$ & $H_1$ group is cyclic with generator $[H]$ induced by $\mathbb{C}P^{n-1} \to \mathbb{C}P^n$ & $[X_F(\mathbb{C})] = d \cdot [H]$.

We can recover $d$ from the intrinsic geometry of $X_F(\mathbb{C})$ using its "Chern classes" in $H^*(X_F(\mathbb{C}))$.

Over the real numbers, $H_{n-1}(\mathbb{R}P^n; \mathbb{Z})$ is cyclic with generator $[H]$ & $[X_F(\mathbb{R})] = d \cdot [H]$

so we recover $d \mod 2$.

It's possible to show that $X_F(\mathbb{R})$ does not provide an upper bound for $d$. 
From a different point of view, the Nash embedding theorem shows that any smooth, closed manifold over \( \mathbb{R} \) is diffeomorphic to \( \mathbb{R}^n \) for some real, smooth, projective variety. For complex manifolds, the analogous statement is false.

To be diffeomorphic to a complex projective variety, a manifold must be complex, Kähler, Hodge, & then it will have an embedding into \( \mathbb{C}P^n \) and Chow's theorem guarantees that it's algebraic. This is what we will spend the bulk of the semester studying. First we'll show that compact submanifolds of \( \mathbb{C}P^n \) satisfy these properties & then that any manifold satisfying these properties is a complex projective variety.